Oct 21st, 12:00 AM

Positive Mass Theorem in All Dimensions

Abdulai Gassama
Clark University, agassama@clarku.edu

Positive Mass Theorem in All Dimensions

An Exploration of General Relativity

Abdulai Gassama
Department of Mathematics, Clark University
LEEP Fellowship 2021
Faculty Advisor: Professor Aghil Alaee

Contact Information:
Email: agassama@clarku.edu

Abstract

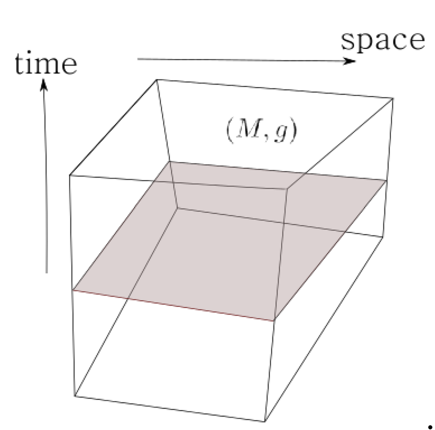
One of the fascinating problems in geometric analysis and mathematical relativity is the positive mass conjecture for spacetime. This problem has been studied extensively for various classes of initial data sets for the Einstein equations. In this project, I study the graphical versions of this problem for asymptotically flat and asymptotically hyperbolic Riemannian manifolds representing black holes, gravitational solitons, and other isolated systems in the Universe.

Main Objectives

1.Study of graphical manifolds over Euclidean space and Hyperbolic space
2. Validate the Positive Mass Theorem of these cases

Introduction

Is the energy/mass of the Universe non-negative? What about stars or black holes? General relativity provides a framework to model objects in large scale. This framework is a $4-$dimensional spacetime which is a Lorentzian manifold with metric (i.e., a quantity to measure size and represent gravitational fields), of which is a solution of the Einstein equations.



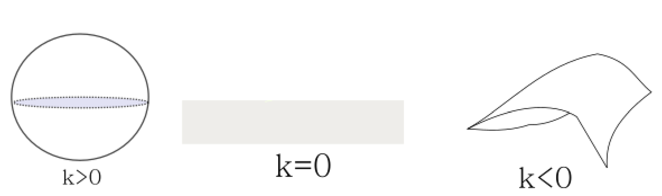
Mass/Energy in General Relativity

The space part of spacetime is modeled by a Riemannian manifold (M, g) where g is the metric (positive definite symmetric metric represented by a matrix).

How to define total energy/mass of spacetime? In Newtonian physics

$$\text{energy} = \int_{\text{Domain}} \text{force}.$$

Hence, what is force? One way to represent it is by the curvature of (M, g). *What* is curvature of a Riemannian manifold? In 2D, we have Gauss curvature k



A notion that involves k that I focused on is the scalar curvature $R(g)$ which is a function of g and its derivative, that is

$$R(g) = F\left(g, \partial g, \partial^2 g\right).$$

For instance, the flat Euclidean space $\mathbb{R}^3$ has metric

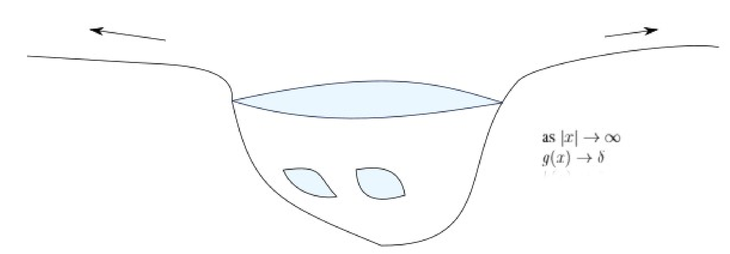
$$\delta = \text{identity matrix}.$$

The curvature of this metric is zero. Conceptually one can say

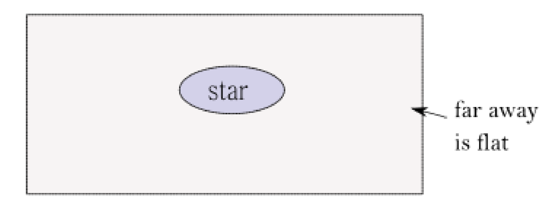
$$\text{energy} \approx \int_{\text{Domain}} \left(R(g) - \underbrace{R_o}_{\text{curvature of model}} \right).$$

Isolated System

How to model an isolated star or a black hole?
We can model these spacetimes with asymptotically flat Riemannian Manifolds.



,



For these manifolds, metric approaches the flat metric near infinity. That means $g = \delta + \underbrace{O\left(r^{-(n-2)}\right)}_{\text{decay}}$ near infinity. The metric is curved inside any compact regions far from infinity.

ADM Mass $\mathbf{m_{ADM}}$

The $m_{ADM}(M, g)$ of an asymptotically flat manifold (M, g) is defined [3] as

$$m_{ADM}(M, g) = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \times \int_{S_r} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j dS_r,$$

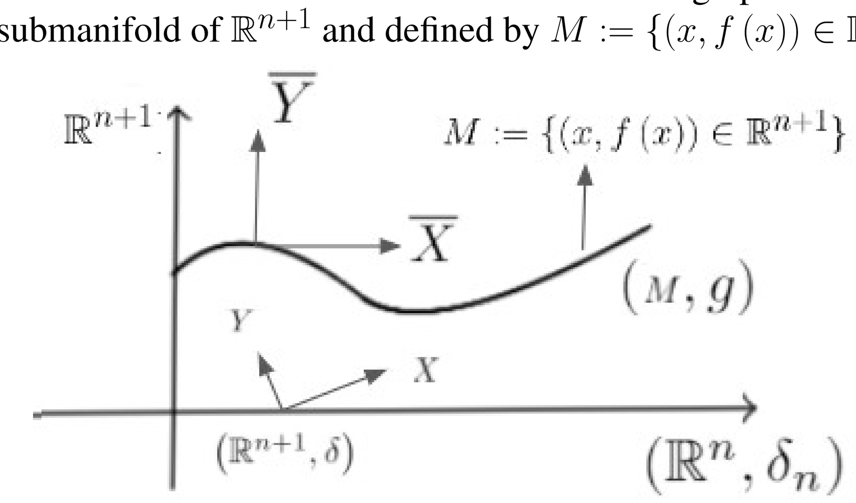
where ω_{n-1} is the volume of the $n-1$ unit sphere, S_r is the coordinate sphere of radius r. ν is the outward unit normal to S_r and dS_r is the area element of S_r in coordinate chart [1].

Graphical and Spacelike Hypersurfaces

The ADM mass of a graph from [3] is

$$m_{ADM}(M, g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \frac{1}{1 + |\nabla f|^2} \left(f_{ii} f_j - f_{ij} f_i\right) \nu^j dA,$$

where ∇f is in respective to the flat metric δ in $\mathbb{R}^n$
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth manifold function. A graph M over $\mathbb{R}^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$ and defined by $M := \{(x, f(x)) \in \mathbb{R}^{n+1}\}$.



Embedded in Euclidean space

If $M = \mathbb{R}^n$, then the metric on graph is defined as

$$g = \delta + df^2,$$

where $df = \sum \frac{\partial f}{\partial x^i} dx^i$ for x^i coordinate $\mathbb{R}^n$.

Theorem 0.1 (PMT for graphs over Euclidean space). *If (M^n, g) is a complete, asymptotically flat graphical manifold over Euclidean space with non-negative scalar curvature. Then*

$$m_{ADM}(M, g) \geq 0,$$

with $m_{ADM}(M, g) = 0$ if and only if (M^n, g) is isometric to $\mathbb{R}^n$ with the standard flat metric.

Sketch of Proof. First, we compute the scalar curvature $R(g)$. After long computation and cancellation of terms we have

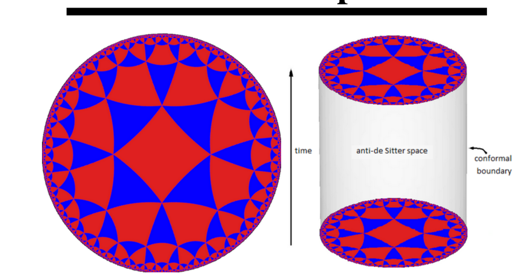
$$R(g) = \frac{1}{1 + |\nabla f|^2} \left((\nabla f)^2 - ||H^f||^2 - \frac{2\nabla f H^2(\nabla f, \nabla f) + 2||H^f(\nabla f, \cdot)||^2}{1 + |\nabla f|^2} \right).$$

This is the divergence of a vector field and we can apply the divergence theorem. Since the manifold is complete, there is only one boundary term at the asymptotically flat manifold. Surprisingly, this leads to the total ADM mass at infinity. In particular, we obtain

$$m_{ADM}(M, g) \geq \int_{\mathbb{R}^n} \frac{R(g)}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

Since $R(g) \geq 0$ then $m_{ADM}(M, g) \geq 0$.

AdS/CFT Correspondence

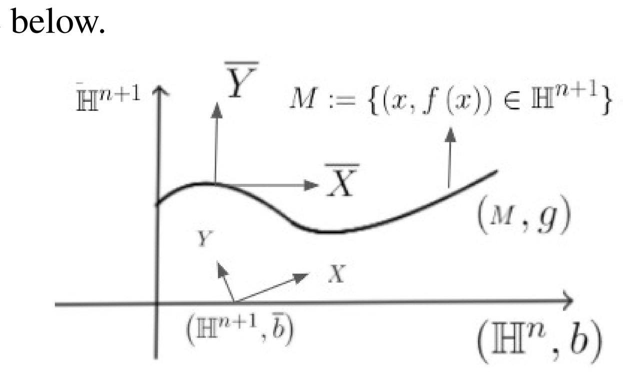


wiki/AdS/CFTcorrespondence

Anti-de-Sitter (AdS) spacetime relates to a hyperbolic spacetime. The related spacetime's energy can be generalized with a negative cosmological constant Λ. Conformal Field Theory(CFT) relates to studying certain symmetries (hence "conformal") that are involved in studying the forces of nature (hence field theory). Ads/CFT correspondence is a useful property that allows for previous calculations in quantum field theory to be solved using works from general relativity and vice versa. The picture above is a visual demonstration of the distance between points in hyperbolic space. Notice that the squares and triangles are bent: in Euclidean space they would not be, and this is primarily due to a negative value for Λ. Space part of spacetime in the AdS are hyperbolic spaces. On the other hand, one can embed n-dimensional hyperbolic space in an $n+1$- dimensional hyperbolic space $\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{H}^n$. In particular, a graph over $\mathbb{H}^n$ is a submanifold $\mathbb{H}^{n+1}$

$$M := \{(x, f(x)) \in \mathbb{H}^{n+1}\}.$$

See the picture below.



Visual of II graph embedded in Hyperbolic Space

A graph over hyperbolic space approaches the hyperbolic space at infinity with a general decay property, $g = b +$ decay term, where b is the canonical metric on $\mathbb{H}^n$. As seen in [2], if $M = \mathbb{H}^n$, then the metric on graph is defined as

$$b = dr^2 + \sinh^2 r\sigma,$$

where σ is the round standard metric of sphere S^{n-1}.

AdS Mass $\mathbf{m_{AH}}$

Let $f : \mathbb{H}^n \to \mathbb{R}$ be a smooth manifold function. Then graph $M = \{(x, f(x)) : x \in \mathbb{H}^{n+1}\}$ is a smooth manifold over $\mathbb{H}^{n+1}$. The induced metric is

$$g = b + v^2 df^2,$$

where b is the canonical metric on $\mathbb{H}^n$ and $v = \sqrt{1 + r^2}$ is the static potential of hyperbolic space $\mathbb{H}^n$. The total mass for an asymptotically hyperbolic Riemannian manifold (M, g) is

$$m_{AH}(M, g) = \lim_{r\to\infty} \int_{S_r} \left(\sqrt{1 + r^2} \left(div^b e - dtr^b e \right) + \left(tr^b e \right) d\sqrt{1 + r^2} - e\left(\nabla^b \sqrt{1 + r^2}, \cdot \right) \right) (\nu_r) dS_r,$$

where $e = g - b$.

Theorem 0.2 (PMT for graphs over Hyperbolic space). *Let (M, g) be an asymptotically hyperbolic graph over $\mathbb{H}^n$. If $R(g) \geq -n(n-1)$ then $m_{AH}(M, g) \geq 0$. The equality holds if and only if (M, g) is isometric to H^n.*

Sketch of Proof. Proof is similar to previous case, but the computation of scalar curvature is more complicated. From [2] finding that the scalar curvature is

$$R(g) = \frac{V^2}{1 + V^2|df|^2} \Bigg[(\Delta f)^2 - |\text{Hess f}|^2 + \frac{2V}{1 + V^2|df|^2} \Big(|\text{Hess f}(\nabla f, \cdot)|^2 - \Delta f \langle \text{Hess f}, df \otimes df \rangle \Big) + \left(2 + \frac{2}{1 + V^2|df|^2} \right) \Delta f \left\langle df, \frac{dV}{V} \right\rangle - \frac{2V^2}{1 + V^2|df|^2} \langle \text{Hess f}, df \otimes df \rangle \left\langle df, \frac{dV}{V} \right\rangle + \frac{2\left\langle df, \frac{dV}{V} \right\rangle^2}{1 + V^2|df|^2} - \frac{2}{1 + V^2|df|^2} |df|^2 |\frac{dV}{V}|^2 - \frac{4}{1 + V^2|df|^2} \langle \text{Hess f}, df \otimes df \rangle \Bigg].$$

I verified that,

$$m_{AH}(M, g) \geq \int_{\mathbb{H}^n} \frac{V\left[R(g) + n(n-1)\right]}{\sqrt{1 + V^2|df|^2}} dV_g.$$

Since $R(g) \geq -n(n-1)$ then $m_{HM}(M, g) \geq 0$.

Future Work

Future research will be on proving a graphical case for a new version of the positive mass theorem, known as the Horowitz-Myers Conjecture.

Horowtiz Myers Geon

The Horowitz-Myers geon (M_{HM}, g_{HM}) is an asymptotically hyperbolic manifold with torus infinity and it has negative mass $m_{HM} < 0$ and maintains the following inequality $R(g) \geq -n(n-1)$.

Graphical Horowitz Myers Conjecture

Conjecture 0.3. *Let (M, g) be a graph over (M_{HM}, g_{HM}) with toroidal infinity. If $R(g) \geq -n(n-1)$ then $m_{AH}(M, g) \geq m_{HM}(M_{HM}, g_{HM})$.*

References

[1] "AdS/CFT Correspondence." Wikipedia, en.wikipedia.org/wiki/AdS/CFT_correspondence, (2020).

[2] A. Sakovich, and R. Gicquaud,and M. Dahl, Penrose Type Inequalities For Asymptotically Hyperbolic Graphs, https://arxiv.org/abs/1201.3321v2, (2012).

[3] M.-K. G. Lam, The Graph Cases of The Riemannian Positive Mass and Penrose Inequalities In All Dimensions, https://arxiv.org/pdf/1010.4256, (2010).