May 17th, 12:00 AM - 11:00 PM

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Voronoi Decomposition of Aperiodic Sets Closed Under Fixed-Parameter Extrapolation

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ABSTRACT
This work explores one facet of an ongoing investigation of the geometric properties and algebraic properties of a family of discrete sets of points in Euclidean space generated by a simple binary operation: pairwise affine combination by a fixed parameter, which is called fixed-parameter extrapolation. These sets display an order and share properties with so-called “quasicrystals” or “quasitilings.” Such sets display some ordered crystal-like properties but are “aperiodic” in the sense that they have no translational symmetry. There are many ways of constructing aperiodic sets and among the most notable of these is via aperiodic tiling of the plane. However, nothing has yet been done to understand how these discrete aperiodic sets correspond to aperiodic tiling. The purpose of the present research is to make the first exploratory, computational steps in this direction.

FIXED-PARAMETER EXTRAPOLATION

Definition 1: Fix a number \( \lambda \in \mathbb{C} \). For any \( a, b, c \in \mathbb{C} \), define \( a\lambda + b \mathbb{R} \). Then for any set \( S \subseteq \mathbb{C} \),
1. If \( |S| = \lambda \), \( \lambda \)-convex iff for every \( a, b \in S \), the point \( a\lambda + b \mathbb{R} \) is in \( S \), and
2. \( S \subseteq \lambda \)-convex closed (or \( \lambda \)-convex for short) iff \( S \subseteq \lambda \)-convex and (topologically) closed.

Definition 2: We define the \( \lambda \)-convex closure of \( S \), denoted \( \lambda \mathcal{R}(S) \), to be the \( \lambda \)-minimum \( \lambda \)-convex superset of \( S \). We let \( R_\lambda \) be shorthand for \( \lambda \mathcal{R}(0, 1) \), the \( \lambda \)-convex closure of \((0, 1) \) [1].

Proposition: If \( 2 < \lambda < 3 \) and \( \lambda = 1 + \varphi \), then \( R_\lambda \) is convex, where \( \varphi = \frac{1 + \sqrt{5}}{2} \), is the golden ratio [1].

RELATION WITH QUASICRYSTALS
When \( \lambda = 1 + \varphi = 2.618 \), \( R_\lambda \) is not convex, but discrete and aperiodic.

\[ R_{1+\varphi} = \left\{ a + b\varphi : a, b \in \mathbb{Z} \right\} \cup \left\{ \frac{a}{\varphi} + b \mathbb{R} : b \in \mathbb{Z} \right\} \]

In particular, except for \( b = 1 \), any two adjacent points of \( R_{1+\varphi} \) differ either by \( \varphi \) or by \( (1 + \varphi) \).

Voronoi Decomposition & Fortune’s Algorithm

Given a set \( P = (p_1, \ldots, p_n) \) of sites, a **Voronoi decomposition** is a subdivision of the space into \( n \) cells, each for one site \( i \in P \) with the property that a point \( q \) lies in the cell corresponding to a site \( p_i \) iff \( d(p_i, q) \leq d(p_j, q) \) for \( i \) distinct from \( j \).

The figure illustrates the fact that \( R_{1+\varphi} \) contains no infinite arithmetic progressions and has no translational symmetry. Thus, \( R_{1+\varphi} \) is a one-dimensional quasiperiodic crystal (quasicrystal).

\[ R_{1+\varphi} \] is a typical example of an aperiodic model set obtained by a cut-and-project scheme, an example of which is given in Figure 1.

**Proposition**: A strong PV number (sPV) is an algebraic integer \( \varphi \) whose Galois conjugates (other than \( \varphi \) and \( \varphi^2 \)) are all in the unit interval \([0, 1) \).

**Theorem**: If \( \varphi \) is a sPV, then \( R_{1+\varphi} \) is discrete.

Given an \( n \)-sided polygon where \( n > 3 \) and odd. The \( \lambda \)-convex closure of the polygon is sometimes discrete (e.g., when \( n = 5, 7, 9, 11 \)).

**Voronoi Decomposition**: From the Voronoi decomposition of these sets, we want to compute a finite set of tiles that correspond to the Voronoi cells of the decomposition. The primary challenge is then modifying Fortune’s algorithm to identify the distinct Voronoi cells that arise, and thereby obtain at least a lower bound on the number of tiles necessary. Ultimately, it is our hope that these computational experiments will deepen our understanding of the relationship between fixed-parameter extrapolation and aperiodic tiles.

**Conclusion**: The purpose of the present research is to make the first exploratory, computational steps in understanding how discrete \( \lambda \)-closed sets correspond to aperiodic tiling. From the Voronoi decomposition of these sets, we want to compute a finite set of tiles that correspond to the Voronoi cells of the decomposition. The primary challenge is then modifying Fortune’s algorithm to identify the distinct Voronoi cells that arise, and thereby obtain at least a lower bound on the number of tiles necessary. Ultimately, it is our hope that these computational experiments will deepen our understanding of the relationship between fixed-parameter extrapolation and aperiodic tiles.