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Voronoi Decomposition of Aperiodic Sets Closed Under Fixed-Parameter Extrapolation

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ABSTRACT

This work explores one facet of an ongoing investigation of the geometric and algebraic properties of a family of discrete sets of points in Euclidean space generated by a simple binary operation: pairwise affine combination by a fixed parameter, which is called fixed-parameter extrapolation. These sets display aperiodic order and share properties with so-called "quasicrystals" or "quasilattices." Such sets display some ordered crystal-like properties but are "aperiodic" in the sense that they have no translational symmetry. There are many ways of constructing aperiodic sets and among the most notable of these is via aperiodic tiling of the plane. However, nothing has yet been done to understand how these discrete aperiodic sets correspond to aperiodic tiling. The purpose of the present research is to make the first exploratory, computational steps in this direction.

FIXED-PARAMETER EXTRAPOLATION

Definition 1: Fix a number $\lambda \in \mathbb{C}$. For any $a, b \in \mathbb{C}$, define $a \star_\lambda b := (1 - \lambda)a + \lambda b$. Then for any set $S \subseteq \mathbb{C}$

1. we say that S is λ-convex iff for every $a, b \in S$, the point $a \star_\lambda b$ is in S, and
2. we say that S is λ-convex closed (or λ-clonvex for short) iff S is λ-convex and (topologically) closed.

where we refer to the operation $\star_\lambda$ as fixed-parameter extrapolation [1].

Definition 2: We define the λ-clonvex closure of S, denoted $R_\lambda(S)$, to be the ⊆-minimum λ-clonvex superset of S. We let R_λ be shorthand for $R_\lambda\{0, 1\}$, the λ-clonvex closure of {0, 1} [1].

Proposition: If $2 < \lambda < 3$ and $\lambda \neq 1 + \varphi$, then R_λ is convex, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio [1].

RELATION WITH QUASICRYSTALS

When $\lambda = 1 + \varphi \approx 2.618$, R_λ is not convex, but discrete and aperiodic.

$$R_{1+\varphi} = \left\{ a + b\varphi : a, b \in \mathbb{Z} \ \& \ \frac{b}{\varphi} \leq a \leq \frac{b}{\varphi} + 1 \right\} = \{1\} \cup \left\{ \left\lceil \frac{b}{\varphi} \right\rceil + b\varphi : b \in \mathbb{Z} \right\}$$

In particular, except for 0 and 1, any two adjacent points of $R_{1+\varphi}$ differ either by φ or by (1 + φ) [1].

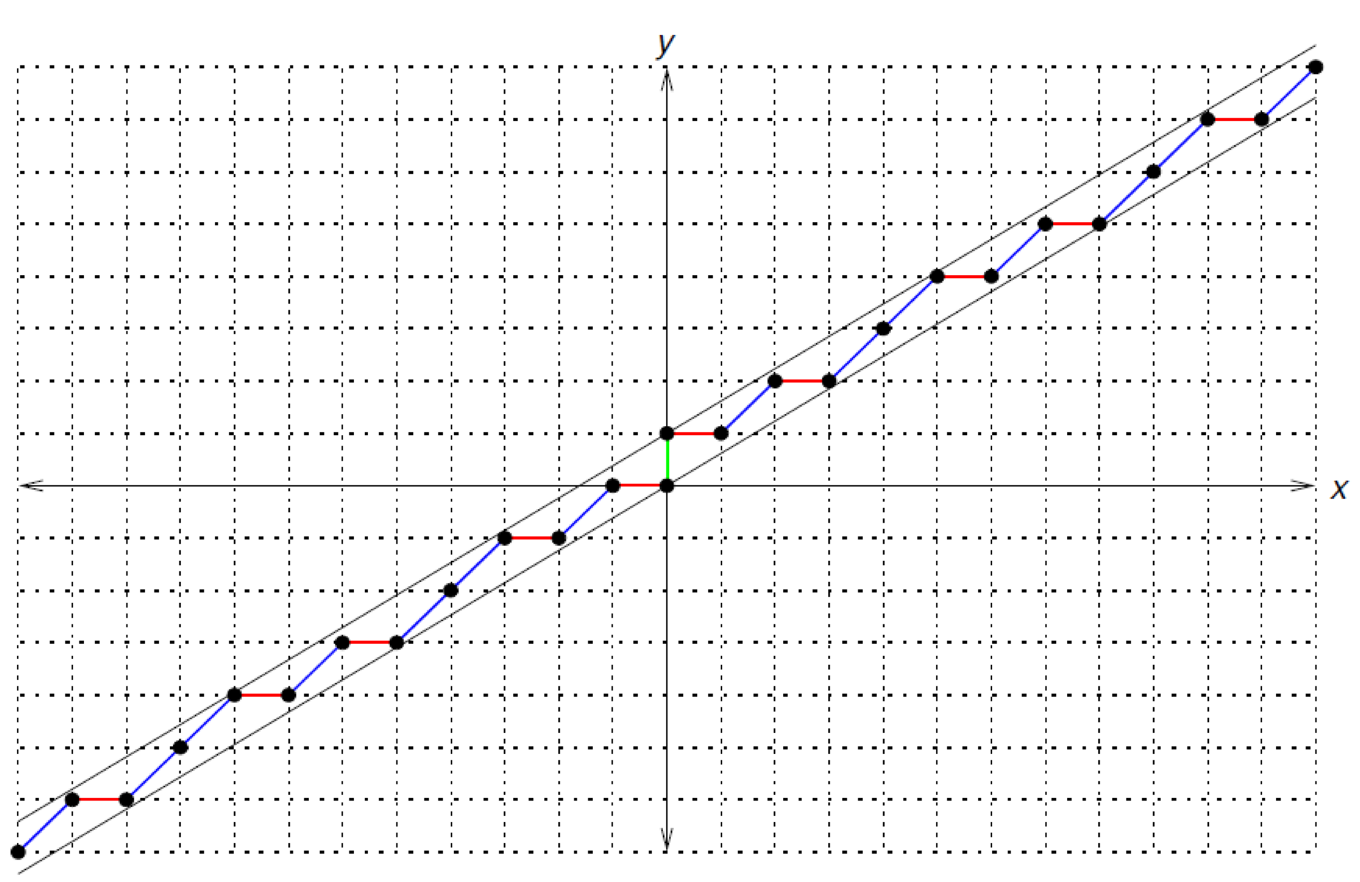

Figure 1: The points $(b, a) \in \mathbb{Z} \times \mathbb{Z}$ such that $a + b\varphi \in R_{1+\varphi}$ are shown. They are all the lattice points lying in the closed strip bounded by the lines $y = x/\varphi$ and $y = x/\varphi + 1$. Except for 0 and 1 (green), distance between neighboring points in $R_{1+\varphi}$ is either φ (red) or 1 + φ (blue)

[1] S. Fenner, F. Green, and S. Homer. Fixed-parameter extrapolation and aperiodic order, 2018.arXiv:1212.2889, version 4, October 2018.

The figure illustrates the fact that $R_{1+\varphi}$ contains no infinite arithmetic progressions and has no translational symmetry. Thus, $R_{1+\varphi}$ is a one-dimensional quasiperiodic crystal (quasicrystal).

$R_{1+\varphi}$ is a typical example of an aperiodic model set obtained by a cut-and-project scheme, an example of which is given in Figure 1.

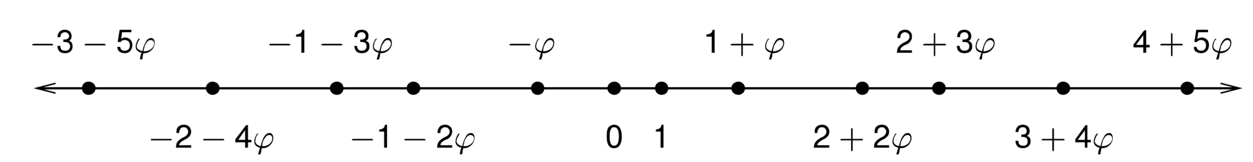

Figure 2: the orthogonal projection of a subset of the lattice points (Figure 1) on the one-dimensional space.

RELATION WITH INFLATION TILING

Definition 3: A strong PV number (sPV) is an algebraic integer α whose Galois conjugates (other than α and α*) are all in the unit interval [0, 1].

Theorem: if λ is a sPV, then R_λ is discrete.

Given an n-sided polygon where n > 3 and odd. The λ-convex closure of the polygon is sometimes discrete (e.g., when n = 5, 7, 9, 13).

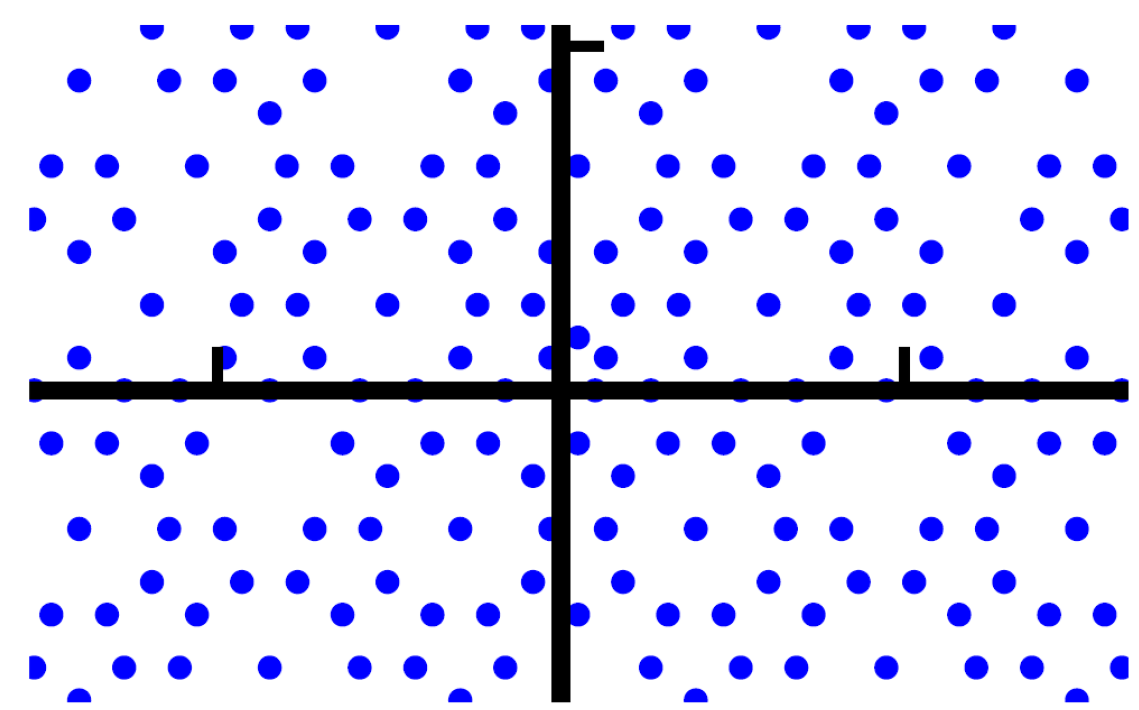

Figure 3: the construction of R_λ given a pentagon (n = 5) as the base, where $\lambda = 1 + \varphi$

Quasicrystals are related to aperiodic tiling of the plane and R_λ is a subset of cut-and-project set, by the theorem above, where λ is a sPV. To further explore how R_λ relate to inflation tiling, we compute the Voronoi decomposition of the calculated set and our key tool is Fortune's Algorithm.

VORONOI DECOMPOSITION & FORTUNE'S ALGORITHM

Given a set $P := \{p_1, \ldots, p_n\}$ of sites, a ***Voronoi decomposition*** is a subdivision of the space into n cells, one for each site in P, with the property that a point q lies in the cell corresponding to a site p_i iff $d(p_i, q) < d(p_j, q)$ for i distinct from j.

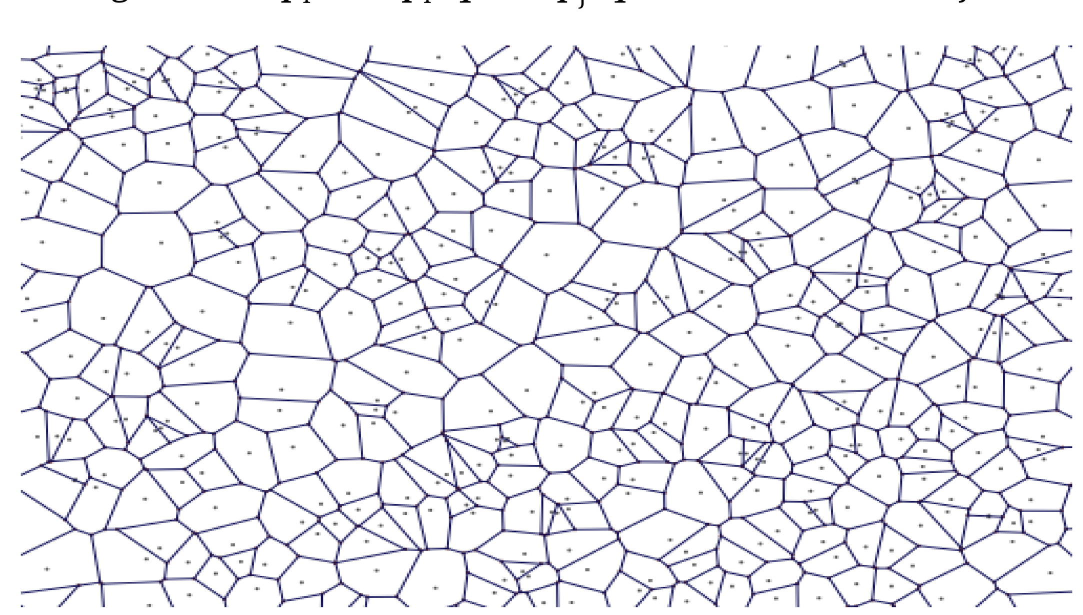

Figure 4: an example of the Voronoi composition of a given set of sites.

Fortune's algorithm is a sweep line algorithm for generating a Voronoi diagram from a set of points in a plane using O(n log n) time and O(n) space. There are two types of event that occur, the site event, and the circle event. The site event happens when the *sweep line* (i.e. a horizontal line that moves from top to bottom) reaches a site, a new parabola will be added to the so-called *beach line* made of arcs of parabolas. The circle event happens when an arc disappears, that is, two neighbor arcs "squeeze" it and a new edge between neighbors is traced out [2].

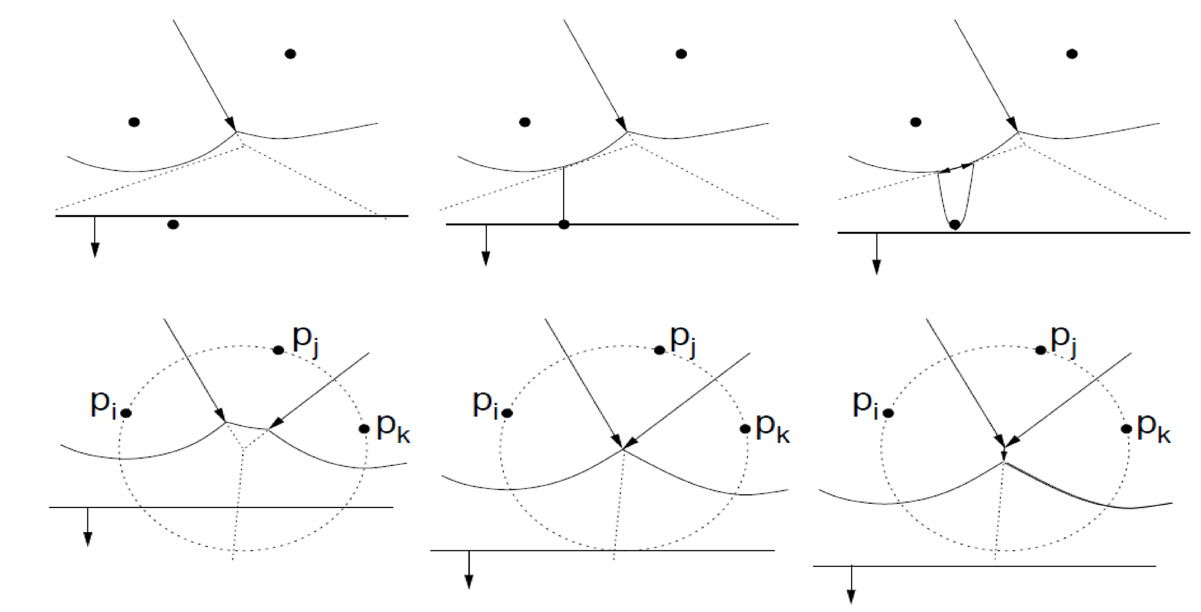

Figure 5: illustration of Fortune's Algorithm. Site event (top) and Circle event (bottom)

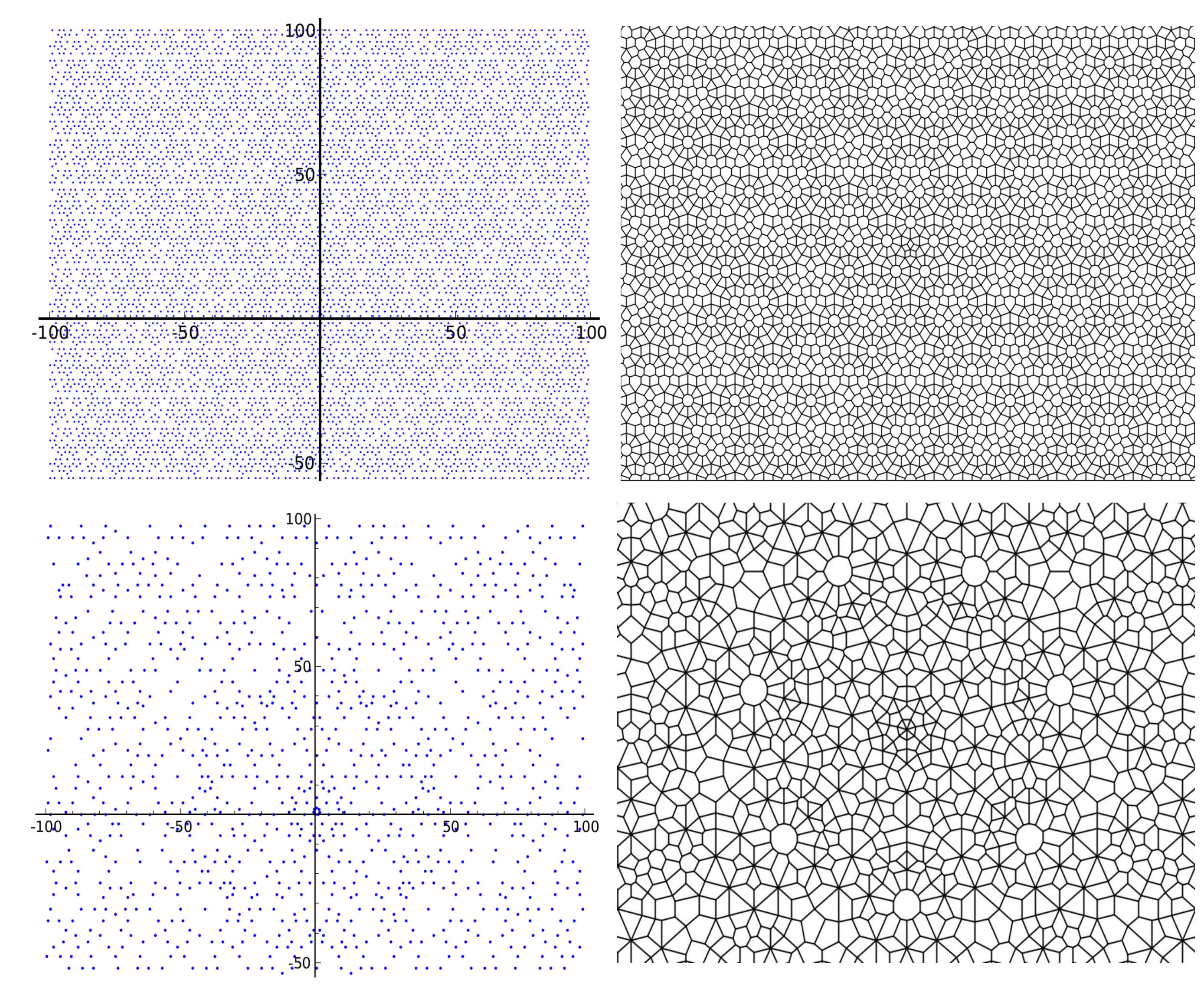

Figure 6: (top) R_λ given a pentagon (n = 5) as the base, where $\lambda = 1 + \varphi$ and its Voronoi diagram (bottom) R_λ given a heptagon (n = 7) as the base, where $\lambda = 5.04892$ and its Voronoi diagram

CONCLUSION

The purpose of the present research is to make the first exploratory, computational steps in understanding how discrete λ-closed sets correspond to aperiodic tiling. From the Voronoi decomposition of these sets, we want to compute a finite set of tiles that correspond to the Voronoi cells of the decomposition. The primary challenge is then modifying Fortune's algorithm to identify the distinct Voronoi cells that arise, and thereby obtain at least a lower bound on the number of tiles necessary. Ultimately, it is our hope that these computational experiments will deepen our understanding of the relationship between fixed-parameter extrapolation and aperiodic tiles.

[2] S. Fortune. A sweepline algorithm for voronoi diargams. Algorithmica, 2:153-174, 1987.