

Clark University

Clark Digital Commons

Mathematics

Faculty Works by Department and/or School

3-2023

Finsler pp-waves and the penrose limit

Amir Babak Aazami

Miguel Ángel Javaloyes

Marcus C. Werner

Follow this and additional works at: <https://commons.clarku.edu/mathematics>



Part of the [Mathematics Commons](#)



Finsler pp-waves and the Penrose limit

Amir Babak Aazami¹ · Miguel Ángel Javaloyes² · Marcus C. Werner³

Received: 25 July 2022 / Accepted: 28 February 2023
© The Author(s) 2023

Abstract

We extend the notion of a Lorentzian pp-wave to that of Finsler spacetimes by providing a coordinate-independent definition of a Finsler pp-wave with respect to the Chern connection; our definition also includes the special case of a plane wave. This treatment introduces suitable lightlike coordinates, in analogy with the Lorentzian case, and utilizes the anisotropic calculus recently developed by one of the authors. We then extend Penrose’s “plane wave limit” to the setting of Finsler spacetimes. New examples of such Finsler pp-waves are also presented.

Keywords pp-waves · Finsler spacetimes · Penrose limit

Contents

1	Introduction	1
2	Preliminaries	1
3	Lightlike coordinates and their properties	1
4	Finsler pp-waves and Brinkmann coordinates	1

MAJ was partially supported by the projects PGC2018-097046-B-I00 and PID2021-124157NB-I00, funded by MCIN/AEI/10.13039/501100011033/ “ERDF A way of making Europe”, and by Ayudas a proyectos para el desarrollo de investigación científica y técnica por grupos competitivos, included in the Programa Regional de Fomento de la Investigación Científica y Técnica (Plan de Actuación 2022) of the Fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia, REF. 21899/PI/22.

✉ Miguel Ángel Javaloyes
majava@um.es

Amir Babak Aazami
aaazami@clarku.edu

Marcus C. Werner
marcus.werner@duke.edu

¹ Department of Mathematics, Clark University, Worcester, MA 01610, USA

² Department of Mathematics, University of Murcia, 30100 Murcia, Spain

³ Zu Chongzhi Center for Mathematics and Computational Sciences, Duke Kunshan University, Kunshan 215316, Jiangsu, China

5	The quotient bundle of a Finsler pp-wave
6	Examples
6.1	Parallel lightlike vector field
6.2	Finsler pp-waves
7	Penrose's construction
8	Concluding remarks
	References

1 Introduction

Among R. Penrose's many accomplishments in general relativity are two foundational results on *plane waves*, a distinguished class of spacetimes modeling radiation propagating at the speed of light. These metrics have a long and rich history within the field of general relativity; they sit inside the more general family of *pp-wave* spacetimes introduced by Ehlers and Kundt [1], which themselves comprise an important subclass of the family of so called *Brinkmann spacetimes* due to H. W. Brinkmann [2], that contain a parallel (covariantly constant) lightlike vector field. A very comprehensive recent survey of plane waves can be found in [3]; pp-waves are now also studied purely in a mathematical context (see, e.g., [4–8]), in addition to their continued usage in gravitational physics (see, e.g., [9, 10]). Of their many properties both physical and mathematical, two of the most noteworthy were discovered by Penrose himself. The first of these, [11], is that plane waves are never globally hyperbolic: this is “remarkable”, to borrow Penrose's own description, all the more so since plane waves are known to be geodesically complete. The second result, [12], no less remarkable, is that every spacetime has a plane wave in a certain well defined limit, a local construction now known as Penrose's “plane wave limit,” perhaps the most important realization of a more general notion of “spacetime limit” due to Geroch [13].

Given the distinguished position that plane waves occupy—and especially in light of Penrose's result that every Lorentzian metric admits one as a limit—it is worthwhile to ask whether there exist analogues of them in *other* geometries of indefinite signature, and, if so, whether Penrose's plane wave limit carries over to such settings as well. A natural direction in which to take this question is that of Finsler geometry of Lorentzian signature, the setting of so called *Finsler spacetimes*, especially in light of their many recent physical applications; see, e.g., [14–20] and the references therein. Indeed, there are already examples of “Finsler pp-waves” in the literature (see, e.g., [21, 22]). In pursuing our question, we are less motivated by the connection between pp-waves, gravitational radiation, and Einstein's equation—indeed, there is no agreed upon analogue of the latter in Finsler geometry—and more by the fact that, via Penrose's limit, plane waves play a central role in Lorentzian geometry per se. What role may they play in Finsler geometry?

In this paper we attempt an answer to this question by first introducing a general definition of Finsler pp-wave, a definition that subsumes the isolated examples of Finsler pp-waves already in the literature; indeed, in Sect. 6 below we introduce additional examples of Finsler pp-waves in accord with our definition. Second, we show that there does exist a notion of “plane wave limit” in the Finslerian setting. These two facts, we argue, make them worthy of study as Finslerian objects. Throughout our

paper, at every step of construction, we carefully present the modifications required to pass from Lorentzian to Finslerian geometry: this includes introducing the Chern connection and its corresponding Chern curvature tensor (Sect. 2), as well as presenting the Finsler analogue of lightlike coordinates (see, e.g., [23]), the Finsler analogue of the invariant definition of pp-wave in terms of the Riemann curvature tensor (see, e.g., [6]), and, finally, the Finsler analogue of the construction of Penrose’s plane wave limit itself. Let us stress that while the notions of lightlike coordinates and parallel lightlike vector fields in Finsler spacetimes depend only on the nonlinear connection, the definition of Finsler pp-waves requires the use of Chern connection. Finally, we note that, while for ease of presentation we have restricted our attention to dimension 4, our definition of Finsler pp-wave extends to all dimensions with only minor modification.

2 Preliminaries

Let M be a (connected) manifold, TM its tangent bundle, and $\pi : TM \rightarrow M$ the natural projection. Let us consider a connected open subset $A \subset TM \setminus \mathbf{0}$ which is conic, namely, satisfying the property that $\lambda v \in A$ for all $v \in A$ and $\lambda > 0$. Let us further assume that A has smooth boundary, that each $A_p = A \cap T_pM$ is nonempty for all $p \in M$, and denote \bar{A} its closure in $TM \setminus \mathbf{0}$. Then, given a function $L : \bar{A} \subseteq TM \rightarrow [0, \infty)$, we say that (M, L) is a *Finsler spacetime* if

- i. L is a smooth function that is positive homogeneous of degree 2 when restricted to each $\bar{A}_p := T_pM \cap \bar{A}$,
- ii. its *fundamental tensor* g , defined for any $v \in \bar{A}$ and $u, w \in T_{\pi(v)}M$ by

$$g_v(u, w) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} L(v + tu + sw) \Big|_{t=s=0}, \tag{1}$$

has signature $(+ - \dots -)$,

- iii. the boundary of \bar{A} in $TM \setminus \mathbf{0}$ coincides with $\mathcal{C} := L^{-1}(0)$, called the *lightcone* of L .

Observe that in this case, \bar{A}_p is convex and salient for every $p \in M$ and the *indicatrix* $\Sigma := L^{-1}(1)$ is a strongly convex hypersurface when restricted to each T_pM ; i.e., each $\Sigma_p = \Sigma \cap T_pM$ is strongly convex. We will say that a vector $v \in TM$ is *timelike* if $v \in A$ and *lightlike* if $v \in \mathcal{C}$, both of which are always non-zero. Observe that we only consider the future cone in the definition of Finsler spacetime, so we will not distinguished future and past causal vectors.

Observe that there are quite a few subtleties in the definition of a Finsler spacetime, but we will adopt the one firstly considered in [24]. Amongst the subtleties to bear in mind we can point out that

- in some definitions, beginning with that of Beem [25], L is defined in the whole TM . Observe that as explained in [14], for an observer $v \in \Sigma$ one can consider g_v as a positive definite metric in its restspace, without any need of considering L defined away from the causal cone.

- In others, some non-smooth directions are allowed [20, 26–28] or they must be smooth up to some power [19] (relevant in the lightlike directions),
- there are others that do not consider the lightlike directions as a part of the model [29],
- there are some models which appear in other contexts with slightly different properties [17, 18].

Associated with the Lorentz-Finsler metric L , there is another anisotropic tensor, usually called the Cartan tensor, defined as

$$C_v(u, w, z) = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} L(v + tu + sw + rz) \Big|_{t=s=r=0}, \tag{2}$$

for any $v \in \bar{A}$ and $u, w, z \in T_{\pi(v)}M$. This symmetric anisotropic tensor is what makes different Finsler spacetimes from the classical Lorentzian geometry. It is straightforward to check that

$$C_v(v, u, w) = C_v(u, v, w) = C_v(u, w, v) = 0, \tag{3}$$

for any $v \in \bar{A}$ and $u, w \in T_{\pi(v)}M$. Moreover, the Levi-Civita–Chern anisotropic connection is a very useful tool for the study of Finsler spacetimes (see [30–32] for more details about anisotropic connections and calculus). Recall that an anisotropic connection can be thought as a connection which depends on directions. This means that for every $v \in \bar{A}$ and $X, Y \in \mathfrak{X}(M)$, one obtains a different value $\nabla_X^v Y \in T_{\pi(v)}M$. In particular, given a chart (\mathcal{U}, φ) , the Christoffel symbols are functions $\Gamma_{ij}^k: \bar{A} \cap T\mathcal{U} \rightarrow \mathbb{R}$ which are homogeneous of degree zero and $\nabla_{\partial_i}^v \partial_j = \Gamma_{ij}^k(v) \partial_k$, where $\partial_1, \dots, \partial_n$ are the partial vector fields of the chart. If we fix a vector field $V \in \mathfrak{X}(\mathcal{U})$ which is \bar{A} -admissible, that is to say, taking values in \bar{A} , we obtain then an affine connection ∇^v in \mathcal{U} with Christoffel symbols $\Gamma_{jk}^i \circ V$. The Levi-Civita–Chern connection can be characterized in terms of the associated affine connections ∇^v . Indeed, it is the only one such that

1. ∇^v is *torsion-free*, namely, $\nabla_X^v Y - \nabla_Y^v X = [X, Y]$ for every $X, Y \in \mathfrak{X}(\mathcal{U})$,
2. ∇^v is *almost g-compatible*, namely

$$X(g_v(Y, Z)) = g_v(\nabla_X^v Y, Z) + g_v(Y, \nabla_X^v Z) + 2C_v(\nabla_X^v V, Y, Z),$$

where $X, Y, Z \in \mathfrak{X}(\mathcal{U})$ and g_v and C_v are the classical tensors obtained when (1) and (2) are evaluated in the vector field V .

Observe that almost g -compatibility is equivalent to $\nabla g = 0$ when this tensor derivative is computed using the anisotropic calculus developed in [30, 31]. Moreover, there is also a Koszul formula that determines ∇^v :

$$\begin{aligned} 2g_v(\nabla_X^v Y, Z) &= X(g_v(Y, Z)) - Z(g_v(X, Y)) + Y(g_v(Z, X)) \\ &\quad + g_v([X, Y], Z) + g_v([Z, X], Y) - g_v([Y, Z], X) \\ 2(-C_v(\nabla_X^v V, Y, Z) - C_v(\nabla_Y^v V, Z, X) + C_v(\nabla_Z^v V, X, Y)). \end{aligned} \tag{4}$$

Let us recall that we say that a causal vector field V is *parallel*, or a *parallel observer* (see [32]) when $\nabla_X^v V = 0$ for all $X \in \mathfrak{X}(\mathcal{U})$. Observe that when V is parallel, then the above Koszul formula coincides with the Koszul formula for g_v . *This means that when V is parallel, the Levi-Civita–Chern connection ∇^v of L coincides with the Levi-Civita connection of g_v —this crucial fact is what enables our definition of Finsler pp-wave in Definition 1 in a manner analogous to that of the Lorentzian version.* Let us say more about this here. To begin with, even if it is not always possible to choose a parallel extension of a vector $v \in \bar{A}$, one can still find a pointwise parallel vector field, as follows: if $p = \pi(v)$, then there exists an extension $V \in \mathfrak{X}(\mathcal{U})$ on a certain neighborhood $\mathcal{U} \subseteq M$ of p such that $(\nabla_X^v V)_p = 0$ for all $X \in \mathfrak{X}(\mathcal{U})$ (see [31, Prop. 2.13]). Indeed, such extensions allow one to easily compute the Chern curvature tensor R_v of L : if $R_v(X, Y)Z$ is the value of the Chern curvature tensor (an anisotropic tensor) for $v \in \bar{A}$ and $X, Y, Z \in \mathfrak{X}(M)$, then, choosing a pointwise parallel extension V of v as above, it follows that

$$R_v(X, Y)Z = R_p^v(X, Y)Z,$$

where R^v is the curvature tensor of ∇^v . The reason for this equality is that, in the general expression of the Chern curvature tensor computed using ∇^v , there are additional tensorial terms apart from R^v of the form $(\nabla_X^v V)_p$ (see [31, Prop. 2.5]) which vanish when V is parallel. So, in this case, the curvature tensor of the Levi-Civita connection of g_v coincides with the Chern curvature tensor R_v evaluated at V .

A function $f : M \rightarrow \mathbb{R}$ admits a gradient with respect to L , denoted ∇f , if there exists a vector field metrically equivalent to df , namely, which satisfies

$$g_{v_f}(\nabla f, X) = df(X)$$

for all $X \in \mathfrak{X}(M)$. Moreover, in this case the Hessian of f is defined as the anisotropic tensor $H^f(X, Y) = \nabla_X df(Y)$, where ∇ is the Levi-Civita–Chern connection of L . Observe that there is a dependence on $v \in A$ in the sense that

$$H_v^f(X, Y) = (\nabla_X df)_v(Y) = X(df(Y)) - df(\nabla_X^v Y).$$

Let us see that H^f is symmetric as

$$H_v^f(X, Y) = X(df(Y)) - df(\nabla_X^v Y) = X(Y(f))(\pi(v)) - \nabla_X^v Y(f),$$

and $\nabla_X^v Y - \nabla_Y^v X = [X, Y]$ (the Levi-Civita–Chern connection is torsion-free), it follows that

$$H_{v_f}^f(X, Y) = g_{v_f}(\nabla_X^{\nabla f}(\nabla f), Y). \tag{5}$$

This follows from the above formula, because using the almost-compatibility of ∇ with g ,

$$\begin{aligned} g_{\nabla f}(\nabla_X^{\nabla f}(\nabla f), Y) &= X(g_{\nabla f}(\nabla f, Y)) - g_{\nabla f}(\nabla f, \nabla_X^{\nabla f} Y) \\ &= X(Y(f)) - \nabla_X^{\nabla f} Y(f) = H_{\nabla f}^f(X, Y), \end{aligned}$$

(the Cartan term vanishes because homogeneity (3)). Let us observe that even if the Hessian is always well-defined for all real functions, the gradient is not. In the classical case of positive definite Finsler metrics, the gradient is well-defined up to the critical points of the function, where it could continuously be extended as zero. In Finsler spacetimes there are more restrictions.

Lemma 1 *A smooth function $f : M \rightarrow \mathbb{R}$ admits a gradient with respect to a Finsler spacetime (M, L) if and only if $df|_A > 0$. Moreover, in this case the gradient is unique.*

Proof For the implication to the right, observe that if ∇f is lightlike at $p \in M$, this means that the kernel of df_p is tangent to \mathcal{C}_p . As A_p is convex, it remains on one side of the hyperplane $\ker(df_p)$, and as $g_{\nabla f}$ is a Lorentzian-type metric (with index $n - 1$) it follows that $df_p|_A > 0$, because the lightlike cone of $g_{\nabla f}$ remains on the same side of $\ker(df_p)$. If ∇f is timelike at $p \in M$, then $\ker(df_p)$ does not touch \bar{A}_p and as $df_p(\nabla f) = g_{\nabla f}(\nabla f, \nabla f) = L(\nabla f) > 0$, it follows that $df_p|_A > 0$.

For the implication to the left, we know that $\ker(df) \cap A = \emptyset$. If $\ker(df_p)$ is tangent to \mathcal{C}_p , then there exists $v \in \mathcal{C}_p \cap \ker(df)$. Then it is easy to see that

$$df_p = \lambda g_v(v, \cdot) = g_{\lambda v}(\lambda v, \cdot) \tag{6}$$

and therefore $\nabla f = \lambda v$. If $\ker(df_p)$ is not tangent to \mathcal{C}_p , then there exists $v \in \Sigma_p$ where the minimum distance between $\ker(df_p)$ and Σ_p is attained. Then (6) holds for some λ . This also shows that the gradient is unique, because the strict convexity of Σ_p implies that the distance is attained in a unique point. \square

Lemma 2 *If a function $f : M \rightarrow \mathbb{R}$ has a gradient field with constant L -norm, then its flow is given by geodesics.*

Proof As the gradient field has constant L -norm, it follows that $X(L(\nabla f)) = 0$ and then

$$0 = 2g_{\nabla f}(\nabla_X^{\nabla f} \nabla f, \nabla f) = 2H_{\nabla f}^f(X, \nabla f) = 2H_{\nabla f}^f(\nabla f, X) = 2g_{\nabla f}(\nabla_{\nabla f}^{\nabla f} \nabla f, X),$$

and as this holds for all $X \in \mathfrak{X}(M)$, it follows that $\nabla_{\nabla f}^{\nabla f} \nabla f = 0$ and ∇f is geodesic. \square

3 Lightlike coordinates and their properties

Although lightlike coordinates exist in all dimensions, we present them here in dimension 4, for convenience. In the following, when a chart (\mathcal{U}, φ) is fixed, we will denote

by $g_{ij} : \bar{A} \cap T\mathcal{U} \rightarrow \mathbb{R}$ the coordinates of the fundamental tensor g in (1), namely, $g_{ij}(v) = g_v(\partial_i, \partial_j)$.

Lemma 3 (*Lightlike coordinates*) *Let (M, L) be a Finsler spacetime and f a smooth function defined on an open subset $\mathcal{U} \subseteq M$. If $N := \nabla f$ is a lightlike vector field, then there exist coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ with $N = \frac{\partial}{\partial \tilde{x}^0}$ about any point in \mathcal{U} in which the metric g_N has components*

$$(g_{ij}(N)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g_{11}(N) & g_{12}(N) & g_{13}(N) \\ 0 & g_{21}(N) & h_{22} & h_{23} \\ 0 & g_{31}(N) & h_{32} & h_{33} \end{pmatrix}, \tag{7}$$

with (h_{ij}) a positive-definite 2×2 matrix.

Proof Of course, “lightlike” means that N is nowhere vanishing yet satisfies $L(N) = 0$; the former implies that the level sets $\mathcal{S}_c := f^{-1}(c)$ are embedded hypersurfaces, while the latter implies that N has geodesic flow, $\nabla_N^N N = 0$, where ∇^N is the Levi-Civita–Chern connection of L (recall Lemma 2). At any $p \in \mathcal{S}_c$, $g_{N_p}(N_p, N_p) = 0$ implies that the induced metric by g_{N_p} on \mathcal{S}_c is degenerate. Let (x^0, x^1, x^2, x^3) denote coordinates within \mathcal{U} in which $N = \partial_0$. Then by the almost compatibility of g and ∇ and the fact that

$$C_N(N, N, \nabla_{\partial_i}^N N) = 0 = C_N(N, \partial_i, \nabla_N^N N),$$

by (3), we have that

$$N(g_N(N, \partial_i)) = g_N(\underbrace{\nabla_N^N N}_0, \partial_i) + g_N(N, \underbrace{\nabla_N^N \partial_i}_{\nabla_{\partial_i}^N N}) = \frac{1}{2} \partial_i(g_N(N, N)) = 0,$$

so that each $c_i := g_{0i}(N)$, $i = 0, 1, 2, 3$, is independent of x^0 . Of course, at least one of these c_i ’s must be *nonzero*, otherwise each N_p^\perp would be four-dimensional. Let us assume that $c_1 \neq 0$. Thus at the moment our metric g_N in the coordinates (x^0, x^1, x^2, x^3) takes the form

$$(g_{ij}(N)) = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & g_{11}(N) & g_{12}(N) & g_{13}(N) \\ c_2 & g_{21}(N) & g_{22}(N) & g_{23}(N) \\ c_3 & g_{31}(N) & g_{32}(N) & g_{33}(N) \end{pmatrix}, \quad c_i = c_i(x^1, x^2, x^3),$$

with the function f satisfying

$$\frac{\partial f}{\partial x^0} = g_N(\nabla f, \partial_0) = 0 \quad , \quad \frac{\partial f}{\partial x^i} = g_N(\nabla f, \partial_i) = c_i. \tag{8}$$

These coordinates, however, are not slice coordinates for the level sets \mathcal{S}_c ; to make them so, simply define new coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ by

$$\begin{cases} \tilde{x}^0 = x^0, \\ \tilde{x}^1 = f(x^1, x^2, x^3), \\ \tilde{x}^2 = x^2, \\ \tilde{x}^3 = x^3. \end{cases}$$

These new coordinates satisfy $\nabla \tilde{x}^1 = \partial/\partial \tilde{x}^0$, and they are indeed slice coordinates for \mathcal{S}_c :

$$\mathcal{S}_c = \left\{ q \in \mathcal{U} : \tilde{x}(q) = \left(\tilde{x}^0(q), c, \tilde{x}^2(q), \tilde{x}^3(q) \right) \right\}.$$

Thus each $N_q^\perp = T_q \mathcal{S}_c = \text{span} \{ \partial/\partial \tilde{x}^0|_q, \partial/\partial \tilde{x}^2|_q, \partial/\partial \tilde{x}^3|_q \}$, with

$$\frac{\partial}{\partial \tilde{x}^0} = \frac{\partial}{\partial x^0} \quad , \quad \frac{\partial}{\partial \tilde{x}^1} \stackrel{(8)}{=} \frac{1}{c_1} \frac{\partial}{\partial x^1} \quad , \quad \frac{\partial}{\partial \tilde{x}^i} \stackrel{(8)}{=} -\frac{c_i}{c_1} \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^i},$$

for $i = 2, 3$, and $\partial/\partial \tilde{x}^0$ has a nice relationship with the other coordinate basis vectors:

$$g_N(\partial/\partial \tilde{x}^0, \partial/\partial \tilde{x}^1) = 1 \quad , \quad g_N(\partial/\partial \tilde{x}^0, \partial/\partial \tilde{x}^i) = -c_i + c_i = 0.$$

The metric in the new coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ is now precisely in the form of (7). Finally, note that since $\partial/\partial \tilde{x}^2$ and $\partial/\partial \tilde{x}^3$ are both orthogonal to the lightlike vector $\partial/\partial \tilde{x}^0$, they must satisfy $g_{22}(N) > 0, g_{33}(N) > 0$. It follows that each embedded 2-submanifold defined by

$$A_{b,c} := \left\{ q \in \mathcal{U} : \tilde{x}(q) = \left(b, c, \tilde{x}^2(q), \tilde{x}^3(q) \right) \right\} \subseteq \mathcal{S}_c, \tag{9}$$

with induced metric $h_{b,c} := \sum_{i=2}^3 g_{ij}(b, c, \tilde{x}^2, \tilde{x}^3, N) d\tilde{x}^i \otimes d\tilde{x}^j$, is Riemannian; indeed, since g_N has negative determinant,

$$\det g_N = -(g_{22}(N)g_{33}(N) - g_{23}(N)^2) < 0 \Rightarrow g_{22}(N)g_{33}(N) - g_{23}(N)^2 > 0. \tag{10}$$

Together with the fact that $g_{22} > 0$, it follows that the two leading submatrices of the 2×2 matrix $\begin{pmatrix} g_{22}(N) & g_{23}(N) \\ g_{32}(N) & g_{33}(N) \end{pmatrix}$ have positive determinant; thus this submatrix is positive definite. Renaming each $g_{ij}(N)$ to h_{ij} for $i, j = 2, 3$, the proof is complete. \square

Lemma 4 *Given an arbitrary lightlike vector v_0 of a Finsler spacetime (M, L) , it is possible to extend it to a lightlike gradient vector field in a certain neighborhood of $\pi(v_0)$.*

Proof First, consider a local splitting $I \times B$ of M with $I \subset \mathbb{R}$ in such a way that $\{t_0\} \times B$ is spacelike for all $t_0 \in I$ and ∂_t is timelike. Then consider a surface S_0 in B diffeomorphic to S^2 which contains $\pi(v_0)$ and is orthogonal to v_0 , with v_0 pointing to the exterior region of $I \times S_0$. Observe that the cylinder $I \times S_0$ is a hypersurface and that it admits a smooth lightlike vector field N along it with the following property: for any $(t_0, p_0) \in I \times S_0$, $N_{(t_0, p_0)}$ is orthogonal to $\{t_0\} \times S_0$. By the Inverse Function Theorem, the exponential map restricted to the bundle generated by N along S_0 is a local diffeomorphism. In particular, one can construct local coordinates around $\pi(v_0)$ using product coordinates in $I \times S_0$ and then the one of the exponential map $s \rightarrow \exp_{(t,p)}(sN)$. All this together implies that the projection onto I in these coordinates provides a function $f: \mathcal{U} \subset M \rightarrow \mathbb{R}$ whose level sets $f^{-1}(t_0)$ are the hypersurfaces obtained as the union of all the lightlike geodesics passing through the points (t_0, p) with $p \in S_0$ and with velocity $N_{(t_0,p)}$. Reducing \mathcal{U} if necessary, we can assume that these hypersurfaces are a piece of the horismos $E^+(\{t_0\} \times S_0) \cup E^-(\{t_0\} \times S_0)$. Therefore, they are degenerate and the gradient vector field ∇f must be lightlike. \square

Observe that the last lemma can be interpreted in the following way. The lightlike gradient vector field can be thought of as the vector field tangent to the light rays departing orthogonally from a given spacelike surface and its temporal cylinder of a given interval of a universal time. In [23, p. 60], there is a sketch of a very similar argument for Lorentzian manifolds.

There is a direct relationship between the domain of validity of lightlike coordinates and the existence of focal points along the geodesic integral curves of N . First, define $\Delta_4 := \sqrt{-\det g_{ij}(N)}$ and

$$\Delta := \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix}^{\frac{1}{2}},$$

and observe that $\Delta_4 = \Delta$. Then the relationship is as follows:

Lemma 5 *In lightlike coordinates, a point $p \in \mathcal{U}$ is a focal point of (9) along the geodesic integral curve of N through p if and only if $\Delta|_p = 0$.*

Proof Let (x^0, x^1, x^2, x^3) be lightlike coordinates as in Lemma 3, with $N = \partial_0$, and let Λ be a Riemannian 2-submanifold as in (9). We begin by observing that $\partial_0, \partial_1, \partial_2, \partial_3$ are all Jacobi fields along any geodesic integral curve $\gamma(x^0)$ of N starting in Λ . Indeed, setting $J := \partial_i$, and noting that $[J, N] = 0$ and $\nabla_N^N N = 0$, we have that

$$J'' = \nabla_N^N \nabla_N^N J = \nabla_N^N \nabla_J^N N = R_N(N, J)N = -R_N(J, N)N,$$

where R_N is the $(1, 3)$ -Chern curvature tensor. Now suppose that $\Delta|_p = 0$ at a point p along γ . Then $\partial_2|_p, \partial_3|_p$ must be linearly dependent, hence some nontrivial linear combination of the two gives the zero vector at p . If we extend this linear combination as is to a vector field $J(x^0)$ along γ , then J will be a Jacobi field. In fact it is a Λ -Jacobi field, since $J(0) \in T_{\gamma(0)}\Lambda$ and since $\tan_{\Lambda} J'(0) = \tan_{\Lambda}(\nabla_{J(0)}^N N)$ (see, e.g., [33, Def. 3.12]); we thus conclude that p is a focal point of Λ along γ . Now for the converse.

Suppose that p is a focal point of Λ along the geodesic integral curve $\gamma(x^0)$ of N , with $\gamma(b) = p$; let $J : [0, b] \rightarrow \mathcal{U}$ denote the corresponding Λ -Jacobi field, with $J(b)$ its (zero) value at p :

$$J(b) = f^i(b)\partial_i|_p = 0.$$

(In fact f^1 is identically zero, since J is orthogonal to γ .) If the $f^i(b)$'s are not all zero, then we are done, since we would thus have a nontrivial linear combination of the $\partial_i|_p$'s equalling zero, which can happen only if $\Delta_4|_p = \Delta|_p = 0$. Thus, assume that each $f^i(b) = 0$, in which case

$$J'(b) = \dot{f}^i(b)\partial_i|_p + \cancel{f^i(b)\nabla_{N(b)}\partial_i} \rightarrow 0$$

where at least one of $\dot{f}^i(b)$ is nonzero, for otherwise $J(b) = J'(b) = 0$ and so J would have to be trivial. Next, because each ∂_i is a Jacobi field, and because any two Λ -Jacobi fields V, W along a geodesic satisfy $g_N(V', W) = g_N(V, W')$ (see, e.g., [33, Prop. 3.18]), we have that

$$\underbrace{g_N(J', \partial_j)|_p}_{\dot{f}^i(b)g_{ij}(N)} = g_N(J, \partial'_j)|_p = 0,$$

the latter because $J(b) = 0$. Thus we've arrived at the system of equations

$$g_{ji}(N_p)\dot{f}^i(b) = 0,$$

with the $\dot{f}^i(b)$'s not all zero. This can only happen if $\Delta_4|_p = 0$, in which case $\Delta_p = 0$ and the proof is complete. □

4 Finsler pp-waves and Brinkmann coordinates

We begin first of all by providing a coordinate-independent definition of a *Finsler pp-wave spacetime* (in dimension 4), analogous to the Lorentzian version to be found in [6]; then, in Theorem 1 below, we show the existence of canonical (Brinkmann) coordinates [2] for any Finsler pp-wave, once again in analogy with the Lorentzian case. In the following, given an A -admissible N defined in some open subset $\mathcal{U} \subset M$, we will use the notation $\Gamma(N^\perp)$ for the space of vector fields $X \in \mathfrak{X}(\mathcal{U})$ such that $g_N(N, X) = 0$.

Definition 1 (pp-waves) Let (M, L) be a 4-dimensional Finsler spacetime with a parallel lightlike gradient vector field N . Then (M, L) is a *pp-wave* if its Chern curvature tensor R_N satisfies

$$R_N(X, Y)Z = 0 \tag{11}$$

for all $X, Y, Z \in \Gamma(N^\perp)$. If, in addition to (11), it is also the case that

$$\nabla_X^N R_N = 0 \tag{12}$$

for all $X \in \Gamma(N^\perp)$, then (M, L) is a *plane wave*.

Remark 1 Let us observe that the definition of parallelism used above depends only on the nonlinear connection associated with L , whose coefficients can be obtained from the Christoffel symbols of the Chern connection as $N_j^i(w) = w^k \Gamma_{kj}^i(w)$. Indeed, in coordinates, $\nabla_X^V V = (X(V^i) + X^j V^k \Gamma_{jk}^i(V))\partial_k = (X(V^i) + X^j N_j^i(V))\partial_k$. However, conditions (11) and (12) are specific to the Chern connection, and using the curvature tensor of the Berwald connection, one would obtain different equations. Recall that the difference between the Chern and Berwald connections can be written in terms of the Landsberg connection, and using [31, Cor. 2.17], one can compute the difference between the curvature tensors of both connections, which is not zero in general. As we will see in Theorem 1, both conditions (11) and (12) admit a very simple characterization in coordinates, which does not seem available for the Berwald connection. This reinforces the Chern connection as the best one to measure the stress-energy tensor in a Finsler spacetime (see [32, 34] for more details).

Before proceeding to a coordinate version of Definition 1, we will first need the following

Lemma 6 *A lightlike gradient vector field $N = \nabla f$ is parallel if and only if all components of the metric (7) are independent of \tilde{x}^0 .*

Proof Assume that $\nabla_X^N N = 0$ for all $X \in \mathfrak{X}(M)$. Consider the partial vectors $N = \partial_0, \partial_1, \partial_2, \partial_3$ of the coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ in Lemma 3, and observe that by (7), we know that $g_N(N, \partial_i) = 0$ if $2 \leq i \leq 3$ and $g_N(N, \partial_1) = 1$. By the Koszul formula (4), taking into account that $N = \partial_0$ and all the Lie Brackets will vanish,

$$0 = 2g_N(\nabla_{\partial_i}^N N, \partial_j) = \partial_i(g_N(N, \partial_j)) - \partial_j(g_N(N, \partial_i)) + N(g_N(\partial_i, \partial_j)),$$

which implies that $N(g_N(\partial_i, \partial_j)) = 0$ for all $i, j = 1, 2, 3$, namely, the coefficients $g_{ij}(N)$ do not depend on the coordinate \tilde{x}^0 .

Assume now that all the $g_{ij}(N)$ do not depend on \tilde{x}^0 . Let us see first that $\nabla_N^N N = 0$. Using the Koszul formula (4),

$$2g_N(\nabla_N^N N, \partial_j) = N(g_N(N, \partial_j)) - \partial_j(g_N(N, N)) + N(g_N(N, \partial_j)) = 0$$

for any $j = 0, 1, 2, 3$, which implies that $\nabla_N^N N = 0$. Using this identity and the Koszul formula once more,

$$2g_N(\nabla_{\partial_i}^N N, \partial_j) = \partial_i(g_N(N, \partial_j)) - \partial_j(g_N(N, \partial_i)) + N(g_N(\partial_i, \partial_j)) = 0,$$

for any $i, j = 0, 1, 2, 3$, which implies that $\nabla_{\partial_i}^N N = 0$. □

Remark 2 Observe that the above lemma implies in particular that a parallel lightlike gradient vector field N is Killing for the Lorentzian metric g_N , but not necessarily for L , as this metric could have a dependence on \tilde{x}^0 . Indeed, we can modify L away from the direction of N without losing the parallel lightlike gradient vector field N , but still losing all the global symmetries.

Theorem 1 (*pp-waves*; [6]) *A 4-dimensional Finsler spacetime (M, L) is a pp-wave if and only if there exist local coordinates (v, u, x, y) in which (7) takes the form*

$$(g_{ij}(N)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & H & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

where $g_{uu}(N) := H(u, x, y)$. Furthermore, (M, L) is a plane wave if and only if $H(u, x, y)$ is quadratic in x, y .

Proof Consider each embedded 2-submanifold $\Lambda_{b,c}$ given by (9), with its corresponding induced Riemannian metric

$$h_{b,c} := \sum_{i,j=2}^3 h_{ij}(b, c, \tilde{x}^2, \tilde{x}^3) d\tilde{x}^i \otimes d\tilde{x}^j.$$

(Since by Lemma 6 each $g_{ij}(N)$ is independent of \tilde{x}^0 , the components h_{ij} are \tilde{x}^0 -independent.) By the Gauss Equation ([35, Theorem 8.5, p. 230]), we know that the (single) component $\text{Rm}_{b,c}(\partial_2, \partial_3, \partial_2, \partial_3)$ of the curvature tensor of $h_{b,c}$ is related to that of g_N by

$$\overbrace{\text{Rm}_N(\partial_2, \partial_3, \partial_2, \partial_3)}^{0, \text{ by (11)}} = \text{Rm}_{b,c}(\partial_2, \partial_3, \partial_2, \partial_3) - g_N(\Pi_{b,c}(\partial_2, \partial_3), \Pi_{b,c}(\partial_3, \partial_2)) + g_N(\Pi_{b,c}(\partial_2, \partial_2), \Pi_{b,c}(\partial_3, \partial_3)), \tag{14}$$

where $\Pi_{b,c}$ is the second fundamental form of $h_{b,c}$; observe that as N is parallel, the curvature tensor of g_N coincides with R_N . But when $N = \partial_0$ is parallel, this equation simplifies considerably, because in such a case the vector field $\nabla_{\partial_2}^N \partial_3$ has no ∂_1 -component,

$$\nabla_{\partial_2}^N \partial_3 = \alpha \partial_0 + \beta \partial_2 + \gamma \partial_3, \tag{15}$$

for some smooth functions α, β, γ (here use that

$$g_N(\nabla_{\partial_2}^N \partial_3, \partial_0) = -g_N(\nabla_{\partial_2}^N \partial_0, \partial_3) = 0$$

because $\partial_0 = N$ is parallel and the compatibility of ∇ and g_N). Thus, since ∂_0 is orthogonal to $\Lambda_{b,c}$,

$$\Pi_{b,c}(\partial_2, \partial_3) := (\nabla_{\partial_2}^N \partial_3)^\perp = \alpha \partial_0, \tag{16}$$

where, as N is parallel, we observe that ∇^N is the Levi-Civita connection of g_N ; likewise for $\Pi_{b,c}(\partial_2, \partial_2)$ and $\Pi_{b,c}(\partial_3, \partial_3)$.¹ Thus, as ∂_0 is lightlike, (14) simplifies to

$$\text{Rm}_{b,c} = 0. \tag{17}$$

In other words, each Riemannian submanifold $(\Lambda_{b,c}, h_{b,c})$ is flat, which means that there exist local coordinates $r_c := r_c(\tilde{x}^2, \tilde{x}^3)$, $s_c := s_c(\tilde{x}^2, \tilde{x}^3)$ on each $\Lambda_{b,c}$ with respect to which the induced metric $h_{b,c}$ takes the form

$$h_{b,c} = dr_c \otimes dr_c + ds_c \otimes ds_c.$$

At each $\tilde{x}^1 = c$, we thus have the triple (c, r_c, s_c) ; considering $-s_c$ if necessary, we may also assume that each $\{\partial_{r_c}, \partial_{s_c}\}$ is positively oriented (namely, $\{\partial_0, \partial_1, \partial_{r_c}, \partial_{s_c}\}$ is positively oriented with the natural orientation of the chart). We now put these together to form a smooth coordinate chart $(\tilde{x}^0, \tilde{x}^1, x, y)$, as follows. First, at each point $(b, c, 0, 0) \in \Lambda_{b,c}$, rotate each $\{\partial_{r_c}, \partial_{s_c}\}$ so that ∂_{r_c} points in the direction of ∂_2 (i.e., $\partial_{r_c} = \frac{1}{\sqrt{h_{22}}} \partial_2|_{(b,c,0,0)}$); these rotated ∂_{r_c} 's thus comprise a smooth vector field $V := \frac{1}{\sqrt{h_{22}}} \partial_2$ on the submanifold $\Sigma := \{(\tilde{x}^0, \tilde{x}^1, 0, 0)\}$. As each ∂_{s_c} is orthogonal to its corresponding ∂_{r_c} and since an orientation has been fixed, the ∂_{s_c} 's thus also comprise a smooth vector field W on Σ , namely, the unique unit length vector field orthogonal to V and such that $\{V, W\}$ is positively oriented. Next, at each $p \in \Sigma \cap \Lambda_{b,c}$, parallel transport V, W along the integral curve $\gamma^{(p)}$ of ∂_{r_c} through p , via the connection $\nabla^{b,c}$ compatible with the induced metric $h_{b,c}$; then, at each point along $\gamma^{(p)}$, parallel transport along the integral curve of ∂_{s_c} on $\Lambda_{b,c}$; by abuse of notation, let V, W denote the resulting vector fields, now smoothly defined on \mathcal{U} . By the Gauss Formula (see, e.g., [35, Theorem 8.2, p. 228]) and the flatness condition (17), we have, on each $(\Lambda_{b,c}, h_{b,c})$, that

$$\nabla_V^N V = \cancel{\nabla_V^{b,c} V} \overset{0}{+} \Pi_{b,c}(V, V);$$

similarly for $\nabla_V^N W$, $\nabla_W^N V$, and $\nabla_W^N W$. But just as in (16), each $\Pi_{b,c}(\cdot, \cdot)$ must point solely in the direction of ∂_0 . Finally, define the orthonormal pair

$$X := -g_N(\partial_1, V)\partial_0 + V \quad , \quad Y := -g_N(\partial_1, W)\partial_0 + W,$$

which pair is now orthogonal to ∂_0, ∂_1 . It follows that all of the Lie brackets of the frame $\{\partial_0, \partial_1, X, Y\}$ will have only a ∂_0 -component, which collectively yields that $X^b = g_N(X, \cdot)$ and $Y^b = g_N(Y, \cdot)$ will be closed; indeed, for any pair of vector fields $A, B \in \{\partial_0, \partial_1, X, Y\}$,

$$dX^b(A, B) = \cancel{A(g_N(X, B))} \overset{0}{-} \cancel{B(g_N(X, A))} \overset{0}{-} \cancel{g_N(X, [A, B])} \overset{0}{=} 0;$$

¹ Since the subspace $S := \text{span}\{\partial_2, \partial_3\}$ is spacelike, its orthogonal complement S^\perp is timelike, and each tangent space $T_p\mathcal{U}$ is the direct sum $S_p \oplus S_p^\perp$ (see, e.g., [36, p. 141]). Therefore, $\alpha\partial_0 \in S^\perp$ is the (unique) normal component of $\nabla_{\partial_2}^N \partial_3 = \alpha\partial_0 + \beta\partial_2 + \gamma\partial_3$.

likewise with dY^b . By the Poincaré Lemma, both 1-forms are locally exact: $X^b = dx$ and $Y^b = dy$, for some smooth functions x, y . With respect to the coordinate chart $(\tilde{x}^0, \tilde{x}^1, x, y)$, the ambient Lorentzian metric g_N thus takes the form

$$(g_{ij}(N)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g_{11}(N) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is precisely (13). (This argument generalizes to dimensions > 4 .) Conversely, suppose that local coordinates (v, u, x, y) exist in which the metric takes this form, with $N = \partial_v = \nabla u$. Then the nonvanishing Christoffel symbols of such a metric g_N are determined by

$$\begin{aligned} \nabla_{\partial_x}^N \partial_u &= \nabla_{\partial_u}^N \partial_x = \frac{H_x}{2} \partial_v, \\ \nabla_{\partial_y}^N \partial_u &= \nabla_{\partial_u}^N \partial_y = \frac{H_y}{2} \partial_v, \\ \nabla_{\partial_u}^N \partial_u &= \frac{H_u}{2} \partial_v - \frac{H_x}{2} \partial_x - \frac{H_y}{2} \partial_y, \end{aligned}$$

where H_u, H_x, H_y denote, respectively, the partial derivatives of H with respect to u, x and y , from which it follows that $R_N(\partial_x, \partial_y)\partial_x = R_N(\partial_x, \partial_y)\partial_y = 0$; this is precisely the curvature condition (11). Actually something else vanishes, too:

$$R_N(\partial_x, \partial_y)\partial_u = \nabla_{\partial_x}^N \nabla_{\partial_y}^N \partial_u - \nabla_{\partial_y}^N \nabla_{\partial_x}^N \partial_u = \frac{H_{yx}}{2} \partial_v - \frac{H_{xy}}{2} \partial_v = 0,$$

so that $R_N(X, Y)W = 0$ for all $X, Y \in \Gamma(N^\perp)$ and $W \in \mathfrak{X}(\mathcal{U})$. Finally, if (12) is also true for all $X \in \Gamma(N^\perp)$, then for any choice $i, j \in \{x, y\}$,

$$\begin{aligned} \underbrace{(\nabla_{\partial_i}^N R_N)(\partial_u, \partial_j, \partial_u)}_0 &= \nabla_{\partial_i}^N (R_N(\partial_u, \partial_j)\partial_u) - R_N(\nabla_{\partial_i}^N \partial_u, \partial_j)\partial_u \\ &= \cancel{\partial_i(\partial_u, \nabla_{\partial_i}^N \partial_j)\partial_u} - \cancel{R_N(\partial_u, \partial_j)\nabla_{\partial_i}^N \partial_u} \\ &= \partial_i(\nabla_{\partial_u}^N \nabla_{\partial_j}^N \partial_u - \nabla_{\partial_j}^N \nabla_{\partial_u}^N \partial_u) - \frac{H_i}{2} \cancel{R_N(\partial_v, \partial_i)\partial_u} \\ &\qquad\qquad\qquad - \frac{H_i}{2} \cancel{R_N(\partial_u, \partial_i)\partial_v} \\ &= \frac{H_{jui}}{2} \cancel{\partial_v} - \frac{H_{ujl}}{2} \cancel{\partial_v} + \frac{1}{2}(H_{xji}\partial_x + H_{yij}\partial_y), \end{aligned}$$

which yields $H_{xji} = H_{yij} = 0$. As the above covariant derivatives of R_N are the only non-vanishing ones up to symmetries with respect to vector fields in $\Gamma(N^\perp)$,

we conclude that $H(u, x, y)$ is quadratic in x, y if and only if $\nabla_X^N R_N = 0$ for all $X \in \Gamma(N^\perp)$, completing the proof of Theorem 1. \square

We will give some examples of Finsler pp-waves in Sect. 6, but let us observe that the examples of Finsler pp-waves already present in literature can be included in the definition of Theorem 1 up to some interpretations. In both cases, [21, 22], the Lorentz-Finsler metric is not smooth on the lightlike parallel vector field N , but as these Finsler spacetimes are of Berwald type, there is an available affine connection which makes N parallel. Moreover, the role of g_N -orthogonality to N can be played by the tangent space to the lightcone at N . In both cases, it coincides with the tangent space to the lightlike cone of a Lorentzian metric which is already a pp-wave with N as parallel vector field. Therefore, the conditions of Theorem 1 are satisfied also for the Berwaldian Finsler pp-waves of [21, 22].

5 The quotient bundle of a Finsler pp-wave

Theorem 2 ([7, 37]) *Let (M, L) be a Finsler spacetime and $N = \nabla f$ a lightlike, parallel gradient vector field defined in an open subset $\mathcal{U} \subseteq M$, with orthogonal complement $N^\perp \subseteq T\mathcal{U}$. Then the vector bundle N^\perp/N admits a positive-definite inner product \bar{g} ,*

$$\bar{g}([X], [Y]) := g_N(X, Y) \quad \text{for all} \quad [X], [Y] \in \Gamma(N^\perp/N),$$

and a corresponding linear connection $\bar{\nabla}: \mathfrak{X}(\mathcal{U}) \times \Gamma(N^\perp/N) \longrightarrow \Gamma(N^\perp/N)$,

$$\bar{\nabla}_W[Y] := [\nabla_W^N Y] \quad \text{for all} \quad W \in \mathfrak{X}(\mathcal{U}) \quad \text{and} \quad [Y] \in \Gamma(N^\perp/N).$$

This connection is flat if and only if $(\mathcal{U}, L|_{\mathcal{U}})$ is a Finsler pp-wave.

Proof The metric \bar{g} will be well defined, and positive definite, whenever N is lightlike; indeed, every $X \in \Gamma(N^\perp)$ not proportional to N is necessarily spacelike, so that \bar{g} is nondegenerate (and positive-definite), and if $[X] = [X']$ and $[Y] = [Y']$, so that $X' = X + fN$ and $Y' = Y + kN$ for some smooth functions f, k , then

$$\bar{g}([X'], [Y']) = g_N(X', Y') = g_N(X, Y) = \bar{g}([X], [Y]).$$

On the other hand, the connection $\bar{\nabla}$ requires N to be parallel or else it is not well defined: $\nabla_W^N Y \in \Gamma(N^\perp)$ if N is parallel, in which case

$$\bar{\nabla}_W[Y'] = [\nabla_W^N Y'] = [\nabla_W^N Y] + \cancel{[W(k)N]} + \cancel{[k\nabla_W^N N]} = \bar{\nabla}_W[Y].$$

That $\bar{\nabla}$ is indeed a linear connection follows easily. Now, if this connection is flat, then by definition its curvature endomorphism

$$\begin{aligned} \bar{R}: \mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U}) \times \Gamma(N^\perp/N) &\rightarrow \Gamma(N^\perp/N), \text{ given by} \\ \bar{R}(U, W)[X] &:= \bar{\nabla}_U[\bar{\nabla}_W[X]] - \bar{\nabla}_W[\bar{\nabla}_U[X]] - \bar{\nabla}_{[U, W]}[X] \end{aligned}$$

vanishes, for any section $[X] \in \Gamma(N^\perp/N)$ and vector fields $U, W \in \mathfrak{X}(\mathcal{U})$. Using the metric \bar{g} , this flatness condition is equivalent to

$$\bar{g}(\bar{R}(U, W)[X], [Y]) = 0 \text{ for all } U, W \in \mathfrak{X}(\mathcal{U}) \text{ , } [X], [Y] \in \Gamma(N^\perp/N).$$

But if we unpack the definitions of $\bar{\nabla}$ and \bar{g} , we see that

$$\bar{g}(\bar{R}(U, W)[X], [Y]) = \text{Rm}_N(U, W, X, Y) = \text{Rm}_N(X, Y, U, W). \tag{18}$$

It follows that $\bar{R} = 0$ if and only if $R_N(X, Y)W = 0$ for all $X, Y \in \Gamma(N^\perp)$ and $W \in \mathfrak{X}(\mathcal{U})$; by (11), this is precisely the condition to be a Finsler pp-wave. \square

6 Examples

6.1 Parallel lightlike vector field

Let us choose a Finsler metric F on $M = \mathbb{R}^2 \times S$ with v, u the coordinates of \mathbb{R}^2 and such that ∂_v is a Killing field, namely, F does not depend on v . Define a one-form ω such that

$$\begin{aligned} \omega(\partial_v) &= F(\partial_v), \\ \omega(\partial_u) &= (1 + g_N(\partial_u, \partial_v))/F(\partial_v), \\ \omega(z) &= g_N(\partial_v, z)/F(\partial_v), \end{aligned}$$

where z is tangent to S and $N := \partial_v$. Then the Lorentz-Finsler metric defined by

$$L(w) := \omega(w)^2 - F(w)^2 \tag{19}$$

for any $w \in TM$, admits N as a lightlike parallel vector field (here we are using the fact that if $A = \{w \in TM : L(w) > 0\}$ is nonempty, then L determines a Finsler spacetime on M , see [16, Theorem 4.1]). To check this, observe that the fundamental tensor g^L of L is given by

$$g_v^L(w, z) = \omega(w)\omega(z) - g_v(w, z)$$

where g is the fundamental tensor of F . By the definition of ω , it follows that the coefficients $g_{ij}^L(N)$, form a matrix as in (7). This implies in particular that N is the L -gradient of the function $f: M = \mathbb{R}^2 \times S \rightarrow \mathbb{R}$ defined as $f(v, u, p) = u$. By Lemma 6, recalling that F has been chosen independent of v and that ω has been constructed from F , we conclude that N is parallel.

6.2 Finsler pp-waves

Let us give a particular example of Finsler pp-wave on $M = \mathbb{R}^2 \times \mathbb{R}^2$. Let (v, u, x, y) be the canonical coordinates of \mathbb{R}^4 and choose a parametric family of Minkowski norms F on \mathbb{R}^2 smoothly depending on the last three coordinates of \mathbb{R}^4 , (u, x, y) . Moreover, define the one-form in \mathbb{R}^2 determined by

$$\begin{aligned} \omega(\partial_v) &= F(\partial_v), \\ \omega(\partial_u) &= (1 + g_N(\partial_u, \partial_v))/F(\partial_v), \end{aligned}$$

where $N = \partial_v$, then

$$L = \omega^2 - F^2 - dx^2 - dy^2 \tag{20}$$

is a Finsler pp-wave defined in the region $A = \{w \in TM : L(w) \geq 0\}$. Observe that the matrix of g_N^L (the fundamental tensor of L), for $N = \partial_v$, is of the form (13). Moreover, this metric is smooth on A , because it is not smooth in the vectors $(0, w) \in T(\mathbb{R}^2 \times \mathbb{R}^2)$ with $w \in T\mathbb{R}^2$ but in this case, $L(0, w) = -w_1^2 - w_2^2 < 0$ if $w \neq 0$. Even if this metric is not of the same type as (19), it is very similar. Indeed, in this case, the role of the metric F in (19) is played by $\sqrt{F^2 + dx^2 + dy^2}$, which is not a regular Finsler metric because of the smoothness issues, but the proof of [16, Theorem 4.1]) still works to guarantee that (20) determines a Finsler spacetime. Moreover, as in the last subsection, one can check that L admits $N = \partial_v$ as a parallel and lightlike vector field, and the coordinates of g_N are of the form (13), which concludes by Theorem 1, that L is a Finsler pp-wave.

7 Penrose’s construction

With the generalizations of lightlike coordinates and pp-waves to Finsler spacetimes in Sects. 3 and 4, respectively, we are now ready to consider the extension of Penrose’s “plane wave limit” [12]. Again, consider a 4-dimensional Finsler spacetime (M, L) and N a lightlike gradient vector field such that locally in lightlike coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$, g_N takes the form

$$(g_{ij}(N)) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g_{11}(N) & g_{12}(N) & g_{13}(N) \\ 0 & g_{21}(N) & h_{22} & h_{23} \\ 0 & g_{31}(N) & h_{32} & h_{33} \end{pmatrix},$$

being $N = \frac{\partial}{\partial \tilde{x}^0}$. Recall that lightlike gradient vector fields always exist at least locally (see Lemma 4). Penrose’s construction now proceeds as follows. Define another coordinate system (x^0, x^1, x^2, x^3) via the diffeomorphism φ given by

$$(\tilde{x}^0, \dots, \tilde{x}^3) \xrightarrow{\varphi} (\tilde{x}^0, \Omega^{-2}\tilde{x}^1, \Omega^{-1}\tilde{x}^2, \Omega^{-1}\tilde{x}^3) := (x^0, \dots, x^3), \tag{21}$$

where $\Omega > 0$ is a constant. Next, define the following metric h_Ω in the new coordinates (x^0, \dots, x^3) ,

$$(h_\Omega)_{ij} := \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \Omega^2 g_{11}(N) & \Omega g_{12}(N) & \Omega g_{13}(N) \\ 0 & \Omega g_{21}(N) & h_{22} & h_{23} \\ 0 & \Omega g_{31}(N) & h_{32} & h_{33} \end{pmatrix}}_{\text{defined in the coordinates } (x^0, x^1, x^2, x^3)},$$

where each component $(h_\Omega)_{ij}$ is (strategically) defined as follows,

$$(h_\Omega)_{11}(x^0, \dots, x^3) := \underbrace{\Omega^2 g_{11}(N)(x^0, \Omega^2 x^1, \Omega x^2, \Omega x^3)}_{= g_{11}(N)(\tilde{x}^0, \dots, \tilde{x}^3)}, \tag{22}$$

$$(h_\Omega)_{22}(x^0, \dots, x^3) := \underbrace{h_{22}(x^0, \Omega^2 x^1, \Omega x^2, \Omega x^3)}_{= h_{22}(\tilde{x}^0, \dots, \tilde{x}^3)}, \tag{23}$$

and similarly with the others. Note that as $\Omega \rightarrow 0$,

$$\lim_{\Omega \rightarrow 0} (h_\Omega)_{11} \stackrel{(22)}{=} 0 \cdot g_{11}(x^0, 0, \dots, 0) = 0,$$

$$\lim_{\Omega \rightarrow 0} (h_\Omega)_{22} \stackrel{(23)}{=} h_{22}(x^0, 0, \dots, 0);$$

similarly with the others. The crucial fact is the following:

Lemma 7 *The metric h_Ω is conformal to $(\varphi^{-1})^* g_N$.*

Proof Use the fact that

$$((\varphi^{-1})^* g_N)(\partial_{x^i}, \partial_{x^j}) dx^i \otimes dx^j = g_N(\partial_{\tilde{x}^i}, \partial_{\tilde{x}^j}) d\tilde{x}^i \otimes d\tilde{x}^j,$$

as well as (21), to obtain

$$\begin{aligned} d\tilde{x}^0 \otimes d\tilde{x}^1 &= \Omega^2 dx^0 \otimes dx^1, \\ g_{11}(N)(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) d\tilde{x}^1 \otimes d\tilde{x}^1 &\stackrel{(22)}{=} \Omega^2 (h_\Omega)_{11}(x^0, x^1, x^2, x^3) dx^1 \otimes dx^1, \\ g_{12}(N)(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) d\tilde{x}^1 \otimes d\tilde{x}^2 &= \Omega^2 (h_\Omega)_{12}(x^0, x^1, x^2, x^3) dx^1 \otimes dx^2, \\ h_{22}(N)(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) d\tilde{x}^2 \otimes d\tilde{x}^2 &\stackrel{(23)}{=} \Omega^2 (h_\Omega)_{22}(x^0, x^1, x^2, x^3) dx^2 \otimes dx^2, \end{aligned}$$

and so on, which clearly yields the relationship

$$(\varphi^{-1})^* g_N = \Omega^2 h_\Omega. \tag{24}$$

This completes the proof. □

In particular, setting $\tilde{g} := (\varphi^{-1})^*g_N$, observe that the homothety (24) means that the Levi-Civita connections of h_Ω and \tilde{g} are equal: $\nabla^\Omega = \nabla^{\tilde{g}}$. Taking the limit

$$\lim_{\Omega \rightarrow 0} h_\Omega = \lim_{\Omega \rightarrow 0} \frac{(\varphi^{-1})^*g_N}{\Omega^2}$$

thus yields the metric

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & h_{22}(x^0, 0, \dots, 0) & h_{23}(x^0, 0, \dots, 0) \\ 0 & 0 & h_{32}(x^0, 0, \dots, 0) & h_{33}(x^0, 0, \dots, 0) \end{pmatrix}}_{\text{defined in the coordinates } (x^0, \dots, x^3)},$$

which is clearly a metric in so-called Rosen coordinates of Lorentzian geometry, also referred to as Baldwin-Jeffery-Rosen coordinates (e.g. [38]) due to the pioneering paper [39] by Baldwin and Jeffery. So, finally, we need to confirm that this can indeed be interpreted as a Finsler pp-wave in Brinkmann coordinates as discussed before. Following the standard coordinate transformation as described e.g. by Blau and O’Loughlin [40], we introduce a matrix Q such that

$$h_{ij}(x_0)Q_k^i(x^0)Q_l^j(x^0) = \delta_{kl},$$

satisfying the symmetry property

$$h_{ij}\dot{Q}_k^iQ_l^j = h_{ij}Q_k^i\dot{Q}_l^j,$$

with $2 \leq i, j, k \leq 3$ labelling the transverse coordinates, and the dot denoting differentiation with respect to x^0 . Since h_{ij} depends smoothly on x^0 , such a matrix Q can always be found for sufficiently small x^0 (e.g. [9]). Then the transformation

$$\begin{aligned} x^0 &= u, \\ x^1 &= v - \frac{1}{2}h_{ij}\dot{Q}_k^iQ_l^j\hat{x}^k\hat{x}^l, \\ x^i &= Q^i_j\hat{x}^j, \end{aligned}$$

gives rise to the metric in Brinkmann coordinates (v, u, \hat{x}^i) with

$$H = -(h_{ij}\dot{Q}_k^i)\dot{Q}_l^j\hat{x}^k\hat{x}^l$$

as the uu -component of g_N in (13).

Remark 3 Observe that the Finslerian Penrose limit depends only on the initial geodesic γ , the one with all the coordinates zero, except \hat{x}^0 . This can be seen using the covariance characterisation of the Penrose limit in [41]. As a first observation, the

Finslerian Penrose limit is a classical Penrose limit for the Lorentzian metric g_N . Then in [41, Section 2], the authors show that the Penrose limit depends only on the parallel transport and the curvature tensor of g_N along γ . As this parallel transport is the one of ∇^N (which is both the Levi-Civita connection of g_N and the Chern connection of L in the direction N) and the curvature tensor of g_N is R_N , the Chern curvature tensor, it turns out that the Finslerian Penrose limit depends only on γ and not on the considered extension N .

8 Concluding remarks

The first main result of this article is the construction of a lightlike gradient vector field N in Finsler spacetimes (see Lemma 4) yielding a particular chart for g_N , as given in Lemma 3. Secondly, after establishing a condition for N to be parallel, we extend the notion of pp-waves, as given by [6], to Finsler spacetimes in Theorem 1. New examples of such Finsler pp-waves are found in Sect. 6, and we also show their quotient bundle structure in Sect. 5. Finally, Penrose's plane wave limit is adapted to Finsler pp-waves in Sect. 7.

In closing, let us note that Lorentzian spacetimes with a parallel lightlike vector field are also known as *Bargmann manifolds*. These have proven useful for studying kinematical groups, in particular the Carroll group of plane waves (notably in [38]), thus raising the question of the kinematical group structure associated with Finsler pp-waves. The optical properties of Finsler pp-waves may offer another worthwhile avenue for future work, as Lorentzian plane waves are well-known to exhibit some remarkable lensing effects (e.g., [42]). Indeed, it may be interesting to see whether Finsler pp-waves can also be regarded as members of a Kundt class generalized to Finsler spacetimes.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability No datasets were generated or analyzed during the current study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ehlers, J., Kundt, W.: Exact solutions of the gravitational field equations. In: Gravitation: an Introduction to Current Research, pages 49–101. Wiley, (1962)
2. Brinkmann, H.W.: Einstein spaces which are mapped conformally on each other. Math. Ann. **94**(1), 119–145 (1925)
3. Sormani, C., Hill, D.C., Nurowski, P., Bieri, L., Garfinkle, D., Yunes, N.: The mathematics of gravitational waves: a two-part feature. Notices AMS **64**(7), 684–707 (2017)

4. Aazami, A.B., Ream, R.: Almost Kahler metrics and pp-wave spacetimes. *Lett. Math. Phys.* **112**, 84 (2022)
5. Flores, J.L., Sánchez, M.: On the geometry of pp-wave type spacetimes. In: *Analytical and Numerical Approaches to Mathematical Relativity*, pages 79–98. Springer, (2006)
6. Globke, W., Leistner, T.: Locally homogeneous pp-waves. *J. Geom. Phys.* **108**, 83–101 (2016)
7. Leistner, T., Schliebner, D.: Completeness of compact Lorentzian manifolds with abelian holonomy. *Mathematische Annalen* **364**(3–4), 1469–1503 (2016)
8. Flores, J.L., Sánchez, M.: The Ehlers-Kundt conjecture about gravitational waves and dynamical systems. *J. Differ. Equ.* **268**(12), 7505–7534 (2020)
9. Blau, M., Figueroa-O'Farrill, J., Papadopoulos, G.: Penrose limits, supergravity and brane dynamics. *Class. Quantum Gravity* **19**(18), 4753 (2002)
10. Blau, M.: Plane waves and Penrose limits. *Lecture Notes for the ICTP School on Mathematics in String and Field Theory (June 2-13 2003)*, (2011)
11. Penrose, R.: A remarkable property of plane waves in general relativity. *Rev. Modern Phys.* **37**(1), 215 (1965)
12. Penrose, R.: Any space-time has a plane wave as a limit. In: *Differential Geometry and Relativity*, pages 271–275. Springer, (1976)
13. Geroch, R.: Limits of spacetimes. *Commun. Math. Phys.* **13**(3), 180–193 (1969)
14. Bernal, A.N., Javaloyes, M.A., Sánchez, M.: Foundations of Finsler spacetimes from the observers' viewpoint. *Universe* **6**(4), 55 (2020)
15. Hohmann, M., Pfeifer, C., Voicu, N.: Relativistic kinetic gases as direct sources of gravity. *Phys. Rev. D* **101**(2), 024062 (2020)
16. Javaloyes, M.A., Sánchez, M.: On the definition and examples of cones and Finsler spacetimes. *Rev. R. Acad. Cienc. Exactas Fis Nat. Ser. A Mat. RACSAM* **114**(1), 30 (2020)
17. Kostelecký, V.A., Russell, N., Tso, R.: Bipartite Riemann-Finsler geometry and Lorentz violation. *Phys. Lett. B* **716**(3–5), 470–474 (2012)
18. Edwards, B.R., Kostelecký, V.A.: Riemann-Finsler geometry and Lorentz-violating scalar fields. *Phys. Lett. B* **786**, 319–326 (2018)
19. Pfeifer, C., Wohlfarth, M.N.R.: Beyond the speed of light on Finsler spacetimes. *Phys. Lett. B* **712**(3), 284–288 (2012)
20. Lämmerzahl, C., Perlick, V.: Finsler geometry as a model for relativistic gravity. *Int. J. Geom. Methods Mod. Phys.* **15**(1), 1850166 (2018)
21. Fuster, A., Pabst, C.: Finsler pp-waves. *Phys. Rev. D* **94**(10), 104072 (2016)
22. Heefer, S., Pfeifer, C., Fuster, A.: Randers pp-waves. *Phys. Rev. D* **104**(2), 024007 (2021)
23. Penrose, R.: *Techniques of Differential Topology in Relativity*. SIAM, (1972)
24. Javaloyes, M.A., Sánchez, M.: Finsler metrics and relativistic spacetimes. *Int. J. Geom. Methods Mod. Phys.* **11**(9), 1460032 (2014)
25. Beem, J.K.: Indefinite Finsler spaces and timelike spaces. *Can. J. Math.* **22**, 1035–1039 (1970)
26. Aazami, A.B., Javaloyes, M.A.: Penrose's singularity theorem in a Finsler spacetime. *Class. Quantum Gravity* **33**(2), 025003 (2016)
27. Caponio, E., Stancarone, G.: On Finsler spacetimes with a timelike Killing vector field. *Class. Quantum Gravity* **35**(8), 085007 (2018)
28. Minguzzi, E.: Affine sphere relativity. *Comm. Math. Phys.* **350**(2), 749–801 (2017)
29. Asanov, G.S.: *Finsler geometry, relativity and gauge theories*. Fundamental Theories of Physics. D. Reidel Publishing Co., Dordrecht (1985)
30. Javaloyes, M.A.: Anisotropic tensor calculus. *Int. J. Geom. Methods Mod. Phys.* **16**(2), 1941001 (2019)
31. Javaloyes, M.A.: Curvature computations in Finsler geometry using a distinguished class of anisotropic connections. *Mediterr. J. Math.* **17**(4), 123 (2020)
32. Javaloyes, M.A., Sánchez, M., Villaseñor, F.F.: Anisotropic Connections and Parallel Transport in Finsler Spacetimes. In: Albuje, A.L., Caballero, M., García-Parrado, A., Herrera, J., Rubio, R. (eds.) *Developments in Lorentzian Geometry*, pp. 175–206. Springer International Publishing, Cham (2022)
33. Javaloyes, M.A., Soares, B.L.: Geodesics and Jacobi fields of pseudo-Finsler manifolds. *Publ. Math. Debrecen* **87**(1–2), 57–78 (2015)
34. Javaloyes, M.A., Sánchez, M., Villaseñor, F.F.: On the significance of the Stress-Energy tensor in Finsler spacetimes. *Universe* **8**(2), 93 (2022)
35. Lee, J.M.: *Introduction to Riemannian Manifolds*, volume 176. Springer, 2nd edition, (2018)

36. O'Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity*, vol. 103. Academic press, (1983)
37. Blanco, O.F., Sánchez, M., Senovilla, J.M.M.: Structure of second-order symmetric Lorentzian manifolds. *J. Eur. Math. Soc.* **15**(2), 595–634 (2013)
38. Duval, C., Gibbons, G.W., Horvathy, P.A., Zhang, P.M.: Carroll symmetry of plane gravitational waves. *Class. Quantum Gravity* **34**(17), 175003 (2017)
39. Baldwin, O.R., Jeffery, G.B.: The relativity theory of plane waves. *Proc. R. Soc. Lond. A* **111**, 95–104 (1926)
40. Blau, M., O'Loughlin, M.: Homogeneous plane waves. *Nucl. Phys. B* **654**, 135–176 (2003)
41. Blau, M., Borunda, M., O'Loughlin, M., Papadopoulos, G.: Penrose limits and spacetime singularities. *Class. Quantum Gravity* **21**(7), 43 (2004)
42. Harte, A.I.: Strong lensing, plane gravitational waves and transient flashes. *Class. Quantum Gravity* **30**, 075011 (2013)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.