3-2023

Prescribing the Q¯ ' -curvature on pseudo-Einstein CR 3-manifolds

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Prescribing the $\overline{Q}'$-Curvature on Pseudo-Einstein CR 3-Manifolds

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December 20, 2022

Abstract In this paper we study the problem of prescribing the $\overline{Q}'$-curvature on embeddable pseudo-Einstein CR 3-manifolds. In the first stage we study the problem in the compact setting and we show that under natural assumptions, one can prescribe any positive (resp. negative) CR pluriharmonic function, if $\int_M Q' \ dv_\theta > 0$ (resp. $\int_M Q' \ dv_\theta < 0$). In the second stage, we study the problem in the non-compact setting of the Heisenberg group. Under mild assumptions on the prescribed function, we prove existence of a one parameter family of solutions. In fact, we show that one can find two kinds of solutions: normal ones that satisfy an isoperimetric inequality and non-normal ones that have a biharmonic leading term.

Keywords: Pseudo-Einstein manifolds, Q'-curvature, Statistical mechanics

2010 MSC. Primary: 32V20, 32V05. Secondary: 82B05

1 Introduction and Main Results

The Q'-curvature and the P'-operator play an important role in the study of the geometry of three-dimensional CR manifolds. In fact, the pair (Q', P') is a close analogue of the pair (Q, P_4) for 4-dimensional conformal manifolds. Indeed, the total Q'-curvature is a biholomorphic invariant while the total Q-curvature in dimension four is tightly linked to the Gauss-Bonnet-Chern formula.

We recall that the Q-curvature was first introduced by Branson [3] in connection with the conformal anomaly of the functional determinant of conformally invariant operators. The total Q-curvature presents an important link between the topology of a manifold and its geometric structure. Indeed, we have

$$\int_M Q \ dv_g + \frac{1}{8}\int_M |W_g|^2 \ dv_g = 4\pi^2\chi(M).$$

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We refer the reader to [17] for some applications of this property. Under conformal change of the metric $g \to \hat{g} = e^{2u}g$, the Q-curvature satisfies

$$e^{4u}Q_{\hat{g}} = Q_g + P_4 u,$$

where P_4 is a fourth order conformally invariant operator with leading term Δ_g^2. We also point out that the pair (Q, P_4) naturally appears in the Beckner-Onofri inequality [1]: for all $u \in H^2(S^4)$,

$$\int_{S^4} uP_4u + 4Qu \, dv_{g_{S^4}} - \Big(\int_{S^n} Q \, dv_{g_{S^4}}\Big) \ln\Big(\frac{\int_{S^4} e^{4u} \, dv_{g_{S^4}}}{Vol(S^4)}\Big) \geq 0.$$

Motivated by these powerful properties for (Q, P_4) and the correspondence between conformal and CR geometry induced by the Fefferman metric [15], one can construct a pair (Q, P_θ) such that under a conformal change of the contact form $\theta \to \hat{\theta} = e^{2u}\theta$, one has

$$P_\theta u + Q_\theta = Q_{\hat{\theta}} e^{4u},$$

where the Paneitz operator is $P_\theta = (\Delta_b)^2 + T^2 + l.o.t.$ The study of this operator led to some strong results related to the embeddability of three-dimensional CR manifolds (see for example [12, 30]). Unfortunately, despite the naturality of this construction, the pair (Q, P_θ) has two main issues. The first one is the size of the kernel of the operator P_4. Indeed, P_4 has a huge kernel containing the space of CR pluriharmonic functions $\mathcal{P}$. Moreover, its fundamental solution has a leading term of $(\ln|xy^{-1}|)^2$ (with M seen as locally diffeomorphic to the Heisenberg group $\mathbb{H}^1$ with its standard group structure defined below). The second issue is that the total Q-curvature is always zero [21]. In fact, if the CR structure is embeddable, then there exists a contact form with pointwise vanishing CR Q-curvature as shown in [30]. Therefore, compared to its conformal counterpart, the CR Q-curvature does not provide strong geometric information.

In [4], the authors provide an alternative operator for odd dimensional spheres that we will denote here by P'. This operator was introduced in order to prove a sharp Beckner-Onofri inequality in the CR setting. In fact, this operator was denoted by A'_Q in [4] and referred to as "conditional intertwinor" because the operator intertwines with the conformal automorphisms modulo functions orthogonal to $\mathcal{P}$. In dimension 3, the P'-operator satisfies $P' = 4(\Delta_b)^2 + lot$ and is defined on the space of pluriharmonic functions and the Q'-curvature is defined implicitly so that

$$e^{2u}Q'_{\hat{\theta}} = Q'_\theta + P'_\theta u + \frac{1}{2}P_\theta(u^2).$$

This can be also stated as

$$P'_\theta u + Q'_\theta = Q'_{\hat{\theta}} e^{2u} \mod \mathcal{P}^\perp. \tag{1}$$

This construction was then extended in [9] to the case of pseudo-Einstein three-dimensional CR manifolds. The total Q'-curvature is invariant under the conformal change of the contact form (within the class of pseudo-Einstein contact forms, or when the CR manifold is assumed to be embeddable and it is only evaluated at Q-flat contact forms, see [30]). In contrast to the CR Q-curvature, the total Q'-curvature is not always zero. In fact, it is proportional to the Burns-Epstein invariant $\mu(M)$ (see [5] when $T^{1,0}M$ is trivial, which was then extended in [13]). In particular, as shown in [9], if (M, J) is the boundary of a strictly pseudo-convex domain X, then

$$\int_M Q' \ \theta \wedge d\theta = 16\pi^2 \Big(\chi(X) - \int_X (c_2 - \frac{1}{3}c_1^2)\Big),$$

where c_1 and c_2 are the first and second Chern forms of the Kähler-Einstein metric on X obtained by solving Fefferman's equation.

Because of the issue of solving orthogonally to the infinite-dimensional space $\mathcal{P}^\perp$, Case, Hsiao and Yang [8], studied another quantity that has similar properties to the Q'-curvature and that comes from the projection of equation (1) on to the space $\mathcal{P}$. In fact, the P'-operator as defined in [4], is only defined after projection on $\mathcal{P}$, but in [8], the authors show extra analytical properties of this projected operator. Indeed, if we let $\Gamma : L^2 \to \mathcal{P}$ be the orthogonal projection and we let $\overline{P}' = \Gamma \circ P'$, then in [8], the authors study the equation

$$\overline{P}' u + \overline{Q}' = \lambda e^{2u} \text{ mod } \mathcal{P}^\perp.$$

The quantity $\overline{Q}'$ is the projection of Q' on $\mathcal{P}$, that is, $\overline{Q}' = \Gamma \circ Q'$.

In this paper we continue the study of the problem of prescribing the $\overline{Q}'$-curvature, under conformal change of the contact structure on pseudo-Einstein CR manifolds. Namely, given a function $Q \in \mathcal{P}$, we want to solve the problem

$$\overline{P}' u + \overline{Q}' = Q e^{2u} \text{ mod } \mathcal{P}^\perp. \tag{2}$$

Naturally, this is equivalent to

$$\overline{P}' u + \overline{Q}' = \Gamma(Q e^{2u}).$$

Notice that if u solves (2), then for $\tilde{\theta} = e^u \theta$, one has $\overline{Q}'_{\tilde{\theta}} = Q$. Indeed, we first observe that the space $\mathcal{P}$ is a CR invariant and does not depend on the contact form, since it can be defined as the set of functions that are locally the real part of $\overline{\partial}_b$-closed complex valued functions. Hence, $\mathcal{P}_\theta = \mathcal{P}_{\tilde{\theta}}$. Moreover, we have two different L^2-inner products. The first one is defined by

$$\langle f, g\rangle_\theta = \int_M fg \ \theta \wedge d\theta,$$

while the second one is defined by

$$\langle f, g\rangle_{\tilde{\theta}} = \int_M fg \ \tilde{\theta} \wedge d\tilde{\theta} = \int_M fg \ e^{2u} \theta \wedge d\theta.$$

Thus, $\psi \in \mathcal{P}_{\theta}^{\perp}$ if and only if $e^{-2u}\psi \in \mathcal{P}_{\tilde{\theta}}^{\perp}$. Therefore, if Γ_u is the L^2-projection using $\langle \cdot, \cdot \rangle_{\tilde{\theta}}$, then we have $\Gamma_u(Q'_{\tilde{\theta}}) = Q$.

Since the method that we are adopting is of a probabilistic nature, we will focus on the case of signed functions. That is, the prescribed function Q will be either positive on M or negative on M. By the invariance of the total Q'-curvature under conformal change, this means that we are focusing on the two cases, $\int_M \overline{Q}' \, dv > 0$ and $\int_M \overline{Q}' \, dv < 0$, respectively. Our main result can be formulated as follow:

Theorem 1.1. *Let $(M, T^{1,0}M, \theta)$ be a three-dimensional embeddable compact pseudo-Einstein manifold such that $\overline{P}'$ is positive and $\ker \overline{P}' = \mathbb{R}$. Consider a function $Q \in C^{\infty}(M)$ such that*

$$\begin{cases} Q > 0 \text{ if } \int_M \overline{Q}' \, dv_\theta > 0 \\ Q < 0 \text{ if } \int_M \overline{Q}' \, dv_\theta < 0. \end{cases}$$

Assume that $\int_M \overline{Q}' \, dv_\theta < 16\pi^2$. Then there exists $u \in \mathcal{P}$ such that

$$P'u + Q' = Qe^{2u} \mod \mathcal{P}^{\perp}.$$

In particular, the contact form $\hat{\theta} = e^u\theta$ satisfies $\overline{Q}'_{\hat{\theta}} = \Gamma_u \circ Q$.

We recall that in [9], the authors show that the non-negativity of the Paneitz operator P_θ and the positivity of the CR-Yamabe invariant imply that $\overline{P}'$ is non-negative and $\ker \overline{P}' = \mathbb{R}$. Moreover, $\int_M Q' \, dv_\theta = \int_M \overline{Q}' \, dv_\theta \leq 16\pi^2$ with equality if and only if $(M, T^{1,0}M, \theta)$ is the standard sphere. In fact, the previously stated assumptions have very strong geometric implications, namely, they imply that the $(M, T^{1,0}M, \theta)$ is embeddable as proved in [12]. We also point out some similarities between our result and the work in [20].

Our strategy follows an idea from statistical mechanics introduced by Messer and Spohn [29], then extended to logarithmic potentials by Kiessling in [24]. This method was used in the problem of prescribing the scalar curvature in [11] and then the problem of prescribing the Q-curvature with conical singularities in [28]. This will be introduced in Section 2.2. In fact, Theorem 1.1 will be a direct corollary of the more general result stated in Theorem 2.5.

In section 4, we consider the case of the Heisenberg group. Since the space is not compact, we will be assuming the following:

Given a function $K \in \ker P' \cap \ker P$ and a positive function $Q \in C^{\infty}(\mathbb{H})$ such that

a) For all $0 < q < 4$, we have $\int_{B_1(x)} \frac{|Q(y)|e^{2K(y)}}{|xy^{-1}|^q} \, dy \to 0$ as $x \to \infty$.

b) There exists $s \geq 0$ such that $\int_{\mathbb{H}} |Q(x)|e^{2K(x)}|x|^s \, dx < \infty$.

Then we have the following result:

Theorem 1.2. *Given a function $Q \in C^\infty(\mathbb{H})$ that satisfies a) and b), there exist a constant $\beta^* < 0$ and a one parameter family u_β, of solutions to*

$$4(\Delta_b)^2 u = Q(x)e^{2u} \ mod\ \mathcal{P}^\perp,$$

where $\beta \in (\beta^*, 0)$ if $Q < 0$ and $\beta \in (0, 8)$ if $Q > 0$. Moreover $u(x) = \frac{1}{2}K(x) - \frac{\beta\gamma}{2}\ln|x| + o(\ln|x|)$.

We recall that the contact form $e^u\theta_0$ is said to be normal, (see [32]), if

$$u(x) = \gamma \int_{\mathbb{H}} \ln \frac{|y|}{|xy^{-1}|} Q(y)e^{2u(y)}\ dy + C,$$

where C is a constant. In particular, if K is not constant in the above theorem, then $e^u\theta_0$ is not normal. Hence, Theorem 1.2 provides us with a families of non-normal contact forms. On the other hand, the results in [32] imply

Corollary 1.3. . *Under the same assumptions as in Theorem 1.1, taking K to be constant, the one parameter family u_β gives rise to contact forms $\theta_\beta = e^{u_\beta}\theta_0$, satisfying the isoperimetric inequality, where θ_0 is the standard contact form on $\mathbb{H}$. That is, for any bounded domain Ω with smooth boundary*

$$Vol_{\theta_\beta}(\Omega) \leq C_\beta Area_{\theta_\beta}(\partial\Omega)^{\frac{4}{3}},$$

where C_β depends on Q and β.

As we will see in Section 4, for K constant, the family of solutions u_β is normal and has total $\overline{Q}'$-curvature equal to $\frac{\beta}{2\gamma}$. Since $\beta < 8$, we have that $\int_{\mathbb{H}} Qe^{2u} < 16\pi^2$, hence, the procedure in [32] can be applied to show that e^{2u} is an A_1 weight. We recall here that a non-negative function w defined on $\mathbb{H}$ is said to be an A_1 weight, if there exists a constant $C_0 > 0$ such that for any ball $B \subset \mathbb{H}$,

$$\frac{1}{|B|}\int_B w(x)\ dx \leq C_0 \inf_{x\in B}(w(x)).$$

Acknowledgement The author wants to express his gratitude to Prof. Paul Yang for the fruitful conversations and insight that helped improve this paper. Also, the author wants to extend his thanks and gratitude to the referees for their comments and suggestions that led to this improved version of the paper.

2 Preliminaries and Setting

2.1 Pseudo-Hermitian geometry

We will closely follow the notations in [9]. Let M^3 be a smooth, oriented three-dimensional manifold. A CR structure on M is a one-dimensional complex subbundle $T^{1,0}M \subset T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ such that $T^{1,0}M \cap T^{0,1}M = \{0\}$ for $T^{0,1}M := \overline{T^{1,0}M}$. Let $H = Re\Big(T^{1,0}M \oplus T^{0,1}M\Big)$ and let $J\colon H \to H$ be the almost complex structure defined by $J(Z+\bar{Z}) = i(Z-\bar{Z})$, for all $Z \in T^{1,0}M$. Notice that since TM and H are both orientable, the line bundle $E = TM/H$ is also orientable and hence trivial. Thus, there exists a globally defined 1-form θ such that $H = \ker\theta$. This form 1-form θ always exists, and is determined up to multiplication by a positive real-valued smooth function. We say that $(M^3, T^{1,0}M)$ is strictly pseudo-convex if the Levi form $d\theta(\cdot, J\cdot)$ on $H \otimes H$ is positive definite for some, and hence any, choice of contact form θ. This is equivalent to the fact that θ is a contact form. We recall that a 1-form θ is said to be a contact form if $\theta \wedge d\theta$ is a volume form on M^3. We shall always assume that our CR manifolds are embeddable and strictly pseudo-convex.

Notice that in a CR-manifold, there is no canonical choice of the contact form θ. A pseudohermitian manifold is a triple $(M^3, T^{1,0}M, \theta)$ consisting of a CR manifold and a contact form. The Reeb vector field T is the vector field such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. The choice of θ induces a natural L^2-dot product $\langle\cdot,\cdot\rangle$, defined by

$$\langle f, g\rangle = \int_M f(x)g(x)\ \theta \wedge d\theta.$$

From now on, we will let $dv_\theta := \theta \wedge d\theta$. We will also use the notation $dv_\theta(x)$ to specify the variable of integration when necessary.

A $(1,0)$-form is a section of $T^*_{\mathbb{C}}M$ which annihilates $T^{0,1}M$. An admissible coframe is a non-vanishing $(1,0)$-form θ^1 in an open set $U \subset M$ such that $\theta^1(T) = 0$. Let $\theta^{\bar{1}} := \overline{\theta^1}$ be its conjugate. Then $d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$ for some positive function $h_{1\bar{1}}$. The function $h_{1\bar{1}}$ is equivalent to the Levi form and without loss of generality, we will normalize our frame so that $h_{1\bar{1}} = 1$. We set $\{Z_1, Z_{\bar{1}}, T\}$ to the dual of $\{\theta^1, \theta^{\bar{1}}, \theta\}$. The geometric structure of a CR manifold is determined by the connection form ${\omega_1}^1$ and the torsion form $\tau_1 = A_{11}\theta^1$ defined in an admissible coframe θ^1 and is uniquely determined by

$$\begin{cases} d\theta^1 = \theta^1 \wedge {\omega_1}^1 + \theta \wedge \tau^1, \\ \omega_{1\bar{1}} + \omega_{\bar{1}1} = dh_{1\bar{1}}, \end{cases}$$

where we use $h_{1\bar{1}}$ to raise and lower indices. That is $\tau^1 = h^{1\bar{1}}\tau_{\bar{1}} = A^1_{\bar{1}}\theta^{\bar{1}}$. The connection forms determine the pseudohermitian connection ∇, also called the Tanaka-Webster connection, by

$$\nabla Z_1 := {\omega_1}^1 \otimes Z_1.$$

The scalar curvature R of θ, also called the Webster curvature, is given by the expression

$$d{\omega_1}^1 = R\theta^1 \wedge \theta^{\bar{1}} \mod \theta.$$

Definition 2.1. *A real-valued function $w \in C^\infty(M)$ is CR pluriharmonic if locally $w = Re(f)$ for some complex-valued function $f \in C^\infty(M, \mathbb{C})$ satisfying $Z_{\bar{1}}f = 0$.*

Equivalently, [27], w is a CR pluriharmonic function if

$$P_3 w := \nabla_1 \nabla_1 \nabla^1 w + i A_{11} \nabla^1 w = 0,$$

for $\nabla_1 := \nabla_{Z_1}$. We denote by $\mathcal{P}$ the space of all CR pluriharmonic functions. Let $\Gamma : L^2(M) \to L^2(M) \cap \mathcal{P}$ be the orthogonal projection on the space of pluriharmonic functions. If $S : L^2(M) \to \ker \bar{\partial}_b$ denotes the Szegő kernel, then

$$\Gamma = S + \bar{S} + F,$$

where F is a smoothing kernel as shown in [23]. The Paneitz operator P_θ is the differential operator

$$\begin{aligned} P_\theta(w) &:= 4\mathrm{div}(P_3 w) \\ &= \Delta_b^2 w + T^2 - 4\mathrm{Im}\nabla^1 \left(A_{11}\nabla^1 f\right), \end{aligned}$$

for $\Delta_b := \nabla^1 \nabla_1 + \nabla^{\bar{1}} \nabla_{\bar{1}}$ the sublaplacian. In particular, $\mathcal{P} \subset \ker P_\theta$. Hence, $\ker P_\theta$ is infinite-dimensional. For a thorough study of the analytical properties of P_θ and its kernel, we refer the reader to [23, 6, 8]. The CR covariance is the main property of the Paneitz operator P_θ needed in this article [21]. That is, if $\hat{\theta} = e^w \theta$, then $e^{2w} P_{\hat{\theta}} = P_\theta$. Other properties — e.g. that it has closed range on embeddable manifolds — are equally important for most problems of the type considered here.

Definition 2.2. *Let $(M^3, T^{1,0}M, \theta)$ be a pseudohermitian manifold. The Paneitz type operator $P'_\theta \colon \mathcal{P} \to C^\infty(M)$ is defined by*

$$\begin{aligned} P'_\theta f = {}& 4\Delta_b^2 f - 8\mathrm{Im}\left(\nabla^1(A_{11}\nabla^1 f)\right) - 4\mathrm{Re}\left(\nabla^1(R\nabla_1 f)\right) \\ &+ \frac{8}{3}\mathrm{Re}(\nabla_1 R - i\nabla^1 A_{11})\nabla^1 f - \frac{4}{3} f \nabla^1(\nabla_1 R - i\nabla^1 A_{11}), \end{aligned} \tag{3}$$

for $f \in \mathcal{P}$.

The main property of the operator P'_θ is its Q-like conformal transformation law as shown in [9]. That is if $(M^3, T^{1,0}M, \theta)$ is a pseudohermitian manifold, $w \in C^\infty(M)$, and we set $\hat{\theta} = e^w \theta$, then

$$e^{2w} \hat{P}'_\theta(u) = P'_\theta(u) + P_\theta(uw), \tag{4}$$

for all $u \in \mathcal{P}$. In particular, since P_θ is self-adjoint and $\mathcal{P} \subset \ker P_\theta$, we have that the operator P' is conformally covariant mod $\mathcal{P}^\perp$.

Definition 2.3. *A pseudohermitian manifold $(M^3, T^{1,0}M, \theta)$ is pseudo-Einstein if $\nabla_1 R - i\nabla^1 A_{11} = 0$.*

Moreover, if θ is a pseudo-Einstein structure then $e^u\theta$ is pseudo-Einstein if and only if $u \in \mathcal{P}$. The definition above was stated in [9], but it was implicitly mentioned in [21]. In particular, if $(M^3, T^{1,0}M, \theta)$ is pseudo-Einstein, then P'_θ takes a simpler form:

$$P'_\theta f = 4\Delta_b^2 f - 8\mathrm{Im}\left(\nabla^1(A_{11}\nabla^1 f)\right) - 4\mathrm{Re}\left(\nabla^1(R\nabla_1 f)\right).$$

Definition 2.4. *Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Einstein manifold. The Q'-curvature is the scalar quantity defined by*

$$Q'_\theta = 2\Delta_b R - 4|A|^2 + R^2, \tag{5}$$

where $|A|^2 = A_{11}A^{\bar{1}\bar{1}}$.

The main equation that we will be dealing with is the change of the Q'-curvature under conformal change. Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Einstein manifold, let $w \in \mathcal{P}$, and set $\hat{\theta} = e^w\theta$. Hence $\hat{\theta}$ is pseudo-Einstein. Then [9]

$$e^{2w}Q'_{\hat{\theta}} = Q'_\theta + P'_\theta(w) + \frac{1}{2}P_\theta\left(w^2\right). \tag{6}$$

In particular, Q'_θ behaves as the Q-curvature for P'_θ, mod $\mathcal{P}^\perp$. To summarize the similarities between the 3-dimensional pseudo-Einstein manifolds and 4-dimensional Riemannian manifolds, we present the following table:

Conformal 4- manifold	Pseudo-Einstein 3-manifold
(M^4, g)	(M^3, θ, J)
$e^{2u}g$	$e^u\theta$; u CR pluriharmonic
$P_g = \Delta_g^2 + \mathrm{div}(\frac{2}{3}R - 2Ric)du$	$P'_\theta = 4\Delta_b^2 - 8\mathrm{Im}(A_{11}u_{\bar{1}})_{\bar{1}} - 4\mathrm{Re}(Ru_1)_{\bar{1}}$
$Q = -\frac{1}{12}(\Delta R - R^2 + 3\|Ric\|^2)$	$Q' = 2\Delta_b R - 4\|A\|^2 + R^2$
$\int_M Q_g + \frac{1}{8}\|W_g\|^2 dv_g = 4\pi^2\chi(M)$	$\int_M Q' dv_\theta = -\frac{\mu(M)}{16\pi^2}$

Since we are working modulo $\mathcal{P}^\perp$ it is convenient to project the previously defined quantities on $\mathcal{P}$. So we define the operator $\bar{P}'_\theta = \Gamma \circ P'_\theta$ and the $\bar{Q}'$-curvature by $\bar{Q}'_\theta = \Gamma(Q'_\theta)$. Notice that

$$\int_M Q' \; \theta \wedge d\theta = \int_M \overline{Q}'_\theta \; \theta \wedge d\theta.$$

Moreover, the operator $\overline{P}'_\theta$ has many interesting analytical properties. Indeed, $\overline{P}'_\theta : \mathcal{P} \to \mathcal{P}$ is an elliptic pseudo-differential operator (see [8]) and if we assume that $\ker \overline{P}'_\theta = \mathbb{R}$, then its Green's function G satisfies

$$\overline{P}'_\theta G(\cdot, y) = \Gamma(\cdot, y) - \frac{1}{V},$$

where $V = \int_M \theta \wedge d\theta$ is the volume of M. Moreover,

$$G(x, y) = -\frac{1}{4\pi^2}\ln(|xy^{-1}|) + \mathcal{K}(x, y),$$

where $\mathcal{K}$ is a bounded kernel as shown in [7]. We recall here the group operation and the Korányi norm in the Heisenberg group $\mathbb{H} = \mathbb{R} \times \mathbb{C}$: If $a = (t_1, z_1)$ and $b = (t_2, z_2) \in \mathbb{H}$, then

$$a \cdot b = (t_1 + t_2 + 2Im(z_1\overline{z_2}), z_1 + z_2).$$

The Korányi norm is defined by

$$|a| = (t_1^2 + |z_1|^4)^{\frac{1}{4}}.$$

We want also to clarify the relation between $\overline{P}'_\theta$ and $\overline{P}'_{\hat{\theta}}$ for $\hat{\theta} = e^u\theta$. Indeed, if we let Γ_u denote the L^2-orthogonal projection on $\mathcal{P}$ induced by $\langle \cdot, \cdot \rangle_{\hat{\theta}}$ then

$$\overline{P}'_{\hat{\theta}} = \Gamma_u \circ (e^{-2u}\overline{P}'_\theta).$$

From now on we will always assume that $\ker \overline{P}' = \mathbb{R}$ and that $\overline{P}'$ is non-negative. We will be using a particular solution, U, to the problem:

$$\overline{P}'U(\cdot, y) = \Gamma(\cdot, y) - \frac{\overline{Q}'}{\int_M \overline{Q}' \ dv_\theta}.$$

One can, then, write $U(x, y) = G(x, y) + H(x) + H(y)$ where G is the Green's function of $\overline{P}'$ and $H \in \mathcal{P}$ is the solution of the problem

$$\overline{P}'H = \frac{1}{V} - \frac{\overline{Q}'}{\int_M \overline{Q}' dv_\theta}.$$

It is easy to check that, locally,

$$U(x, y) = -\gamma \ln |xy^{-1}| + \tilde{\mathcal{H}}(x, y),$$

where $\gamma = \frac{1}{4\pi^2}$.

The proof of Theorem 1.1 will be a direct consequence of the following

Theorem 2.5. *We fix a smooth function Q such that $Q(x) > 0$ on M. For every $\beta < \frac{8}{\gamma}$, there exist $\rho_\beta \in L^p(M)$ for all $1 \leq p < \infty$, solving the following fixed point problem:*

$$\rho_\beta(x) = \frac{Q(x) \exp\left(\beta \int_M U(x, y)\rho_\beta(y) \ dv_\theta(y)\right)}{\int_M Q(x) \exp\left(\beta \int_M U(x, y)\rho_\beta(y) \ dv_\theta(y)\right) \ dv_\theta(x)}.$$

The idea of the proof of the previous result follows a procedure introduced by Messer and Spohn [29] for the a smooth interaction potential. This method was then developed by Kiessling [24, 25, 26]. The method mainly consists of studying the typical distribution of a family of particles inside a set that interact through a given Hamiltonian. In our case it will be U. In order to develop this method, we need some probabilistic background. For the sake of notation, we will write $dv_\theta(x) = dx$ and $dv_\theta(y) = dy$.

2.2 Overview of the Probabilistic Method

We first define the Hamiltonian, or the potential, of N particles in the manifold M. That is, given $N \in \mathbb{N}$ and $x_1, \cdots, x_N \in M$, the Hamiltonian $U^{(N)}$ is defined by

$$U^{(N)}(x_1, \cdots, x_N) = \frac{1}{2(N-1)} \sum_{1 \le i \ne j \le N} U(x_i, x_j) = \frac{1}{N-1} \sum_{1 \le i < j \le N} U(x_i, x_j).$$

We now introduce some probabilistic tools. For each $N \in \mathbb{N}$, denote the probability measures on $M^{(N)} = \overbrace{M \times M \times \cdots \times M}^{N}$ by $P(M^{(N)})$. For a probability measure $\varrho^{(N)} \in P(M^{(N)})$, denote the associated Radon measure by $\hat{\varrho}^{(N)}$ and by this we mean, its action on functions, that is

$$\hat{\varrho}^{(N)}(f) = \int_{M^{(N)}} f(y) \varrho(dy).$$

A measure $\mu^{(N)} \in P(M^N)$ is called absolutely continuous with respect to a measure $\varrho^{(N)} \in P(M^{(N)})$, written $d\mu^{(N)} << d\varrho^{(N)}$, if there exists a positive $d\varrho^{(N)}$-integrable function $f(x_1, ..., x_N)$, called the density of $\mu^{(N)}$ with respect to $\varrho^{(N)}$, such that $d\mu^{(N)} = f(x_1, ..., x_N) d\varrho^{(N)}$. By $P^s(M^{(N)})$ we mean the space of exchangeable probabilities, i.e. the subset of $P(M^{(N)})$ whose elements are permutation symmetric in x_1, ..., $x_N \in M$. The n^{th} marginal measure of $\varrho^{(N)} \in P^s(M^{(N)})$, $n < N$, is an element of $P^s(M^{(n)})$, given by integrating $\varrho^{(N)}$ with respect to $N-n$ variables. More precisely, given a measurable set $A \subset M^n$, the n^{th} marginal $\varrho_n^{(N)}$ is given by

$$\varrho_n^{(N)}(A) = \varrho^{(N)}(A \times M^{(N-n)}).$$

We let $\Omega = M^{(\mathbb{N})}$ be the set of sequences with values in M. To $\varrho \in P(M)$ we assign the energy functional defined by

$$\mathcal{E}(\varrho) \equiv \frac{1}{2} \hat{\varrho}^{\otimes 2}(U(x,y)) = \frac{1}{2} \int_M \int_M U(x,y) \varrho(dx) \varrho(dy), \tag{7}$$

whenever the integral on the right exists. We denote by $P_{\mathcal{E}}(M)$ the subset of $P(M)$ for which $\mathcal{E}(\varrho)$ exists. For $\mu \in P^s(\Omega)$ the mean energy of μ is defined by

$$e(\mu) = \lim_{n \to \infty} \frac{1}{n} \hat{\mu}_n(U^{(n)}) = \frac{1}{2} \hat{\mu}_2(U(x,y)), \tag{8}$$

whenever the integral on the right exists. Using the decomposition measure introduced by [19], one has the following proposition:

Proposition 2.6. *The mean energy of μ, is well defined for those μ whose decomposition measure $\nu(d\varrho|\mu)$ is concentrated on $P_{\mathcal{E}}(M)$, and in that case it is given by*

$$e(\mu) = \int_{P_{\mathcal{E}}(M)} \nu(d\varrho|\mu) \mathcal{E}(\varrho). \tag{9}$$

In our setting, we define the measure

$$\tau(dx) = Q(x)dx, \tag{10}$$

and we set $\mathcal{M}^{(1)} = \int_M Q(y)dy$. Thus one can define the probability measure $\mu^{(1)}(dx) = \frac{1}{\mathcal{M}^{(1)}}\tau(dx)$. Next, we define the micro-canonical ensemble, [14], by

$$\mu^{(N)} = \frac{1}{\mathcal{M}^{(N)}(\beta)} \exp\left(\beta \frac{1}{N-1} \sum_{1\le i<j\le N} U(x_i, x_j)\right) \prod_{1\le l\le N} \tau(dx_l), \tag{11}$$

where $\mathcal{M}^{(N)}(\beta)$ is a normalizing constant making $\mu^{(N)}$ a probability measure. That is

$$\mathcal{M}^{(N)}(\beta) = \int_{M^{(N)}} \exp\left(\beta \frac{1}{N-1} \sum_{1\le i<j\le N} U(x_i, x_j)\right) \prod_{1\le l\le N} \tau(dx_l).$$

For each $\varrho^{(N)}(dx_1...dx_N) \in P\big(M^{(N)}\big)$, its entropy with respect to the probability measure $\mu^{(1)}(dx_1) \otimes ... \otimes \mu^{(1)}(dx_N) \equiv \mu^{(1)\otimes N}(dx_1...dx_N)$ is defined by

$$\mathcal{S}^{(N)}\left(\varrho^{(N)}\right) = -\int_{M^{(N)}} \ln\left(\frac{d\varrho^{(N)}}{d\mu^{(1)\otimes N}}\right) \varrho^{(N)}(dx_1...dx_N), \tag{12}$$

if $\varrho^{(N)}$ is absolutely continuous with respect to $d\tau^{\otimes N}$, and provided the integral exists. In all other cases, $\mathcal{S}^{(N)}\left(\varrho^{(N)}\right) = -\infty$. In particular, if μ_n is the n^{th} marginal of a measure $\mu \in P^s(\Omega)$, then the entropy of μ_n, $n \in \{1, ...\}$, is given by $\mathcal{S}^{(n)}(\mu_n)$, where $\mathcal{S}^{(n)}$ is defined as in (12) with $\varrho^{(n)} = \mu_n$. We also define $\mathcal{S}^{(0)}(\mu_0) = 0$.

After having defined the entropy function, we now state some of its classical properties. We refer the reader to [26] for the details of the proofs. For each $\mu \in P^s(\Omega)$, the sequence $n \mapsto \mathcal{S}^{(n)}(\mu_n)$ enjoys the following

Proposition 2.7. ***Non-positivity***
For all n,

$$\mathcal{S}^{(n)}(\mu_n) \le 0.$$

Monotonic decrease
If $n < n_1$, then

$$\mathcal{S}^{(n_1)}(\mu_{n_1}) \le \mathcal{S}^{(n)}(\mu_n).$$

Strong sub-additivity *For n_1, $n_2 \le n$,*

$$\mathcal{S}^{(n)}(\mu_n) \le \mathcal{S}^{(n_1)}(\mu_{n_1}) + \mathcal{S}^{(n_2)}(\mu_{n_2}) + \mathcal{S}^{(n-n_1-n_2)}(\mu_{n-n_1-n_2}) - \mathcal{S}^{(n_1+n_2-n)}(\mu_{n_1+n_2-n}),$$

with the convention that $\mathcal{S}^{(-m)}(\mu_{-m}) \equiv 0$ for $m > 0$.

As a consequence of the sub-additivity of $\mathcal{S}^{(n)}(\mu_n)$, the limit

$$\mathcal{S}(\mu) = \lim_{n\to\infty} \frac{1}{n}\mathcal{S}^{(n)}(\mu_n)\,,$$

exists whenever $\inf_n n^{-1}\mathcal{S}^{(n)}(\mu_n) > -\infty$; otherwise $\mathcal{S}(\mu) = -\infty$. The quantity $\mathcal{S}(\mu)$ is called the mean entropy of $\mu \in P^s(\Omega)$. The mean entropy is an affine function, moreover one has the following representation .

Proposition 2.8. *The mean entropy of μ, is given by*

$$\mathcal{S}(\mu) = \int_{P(M)} \nu(d\varrho|\mu)\mathcal{S}^{(1)}(\varrho).$$

3 Proof of Theorem 1.1

3.1 First Properties of the Probability Measures

We begin investigating our problem by following the approach developed in [26].

First we have the following integrability property.

Proposition 3.1. *For $\beta\gamma \in (-\infty, 8)$, the measure $\mu^{(N)}$ satisfies $d\mu^{(N)} << d\tau^{\otimes N}$, moreover, the associated density belongs to $L^p(M^{(N)}, \tau^{\otimes N})$ for $p \in [1,\infty]$ if $\beta \le 0$ and $p \in [1, \frac{8}{\beta\gamma})$ if $\beta\gamma \in (0,8)$, for N big enough.*

Proof. Indeed, using the convexity of the exponential function and the symmetry of U, we have

$$\begin{aligned}
\mathcal{M}^{(N)}(p\beta) &= \int_{M^{(N)}} \exp\Big(p\beta U^{(N)}(x_1,\cdots,x_N)\Big) d\tau^{\otimes N}(x_1,\cdots,x_N)\\
&\le \frac{1}{N}\sum_{i=1}^{N}\int_{M^{(N)}} \exp\Big(\frac{p\beta}{2}\frac{N}{N-1}\sum_{j=1,j\neq i}^{N} U(x_i,x_j)\Big) d\tau^{\otimes N}(x_1,\cdots,x_N)\\
&\lesssim \frac{1}{N}\sum_{i=1}^{N}\int_{M}\Big(\int_{M} \exp(\frac{p\beta}{2}\frac{N}{N-1}U(x,y))\tau(dy)\Big)^{N-1}\tau(dx)\\
&\lesssim 1+\int_{M}\Big(\int_{B_x(1)} \exp\Big(\frac{p\beta\gamma}{2}\frac{N}{N-1}\ln(\frac{1}{|xy^{-1}|})\Big)\tau(dy)\Big)^{N-1}\tau(dx).
\end{aligned}$$

It is clear that the integrand is finite, whenever $p\beta\gamma\frac{N}{N-1} < 8$. $\square$

We set the approximated variational problem by defining the functional $\mathcal{F}_\beta^{(N)}$ as follows

$$\mathcal{F}_\beta^{(N)}(\varrho^{(N)}) := \mathcal{S}^{(N)}(\varrho^{(N)}) + \beta\hat{\varrho}^{(N)}\left(U^{(N)}\right).$$

This functional is well defined on probability measures in $P(M^{(N)}) \bigcap \cup_{p>1} L^p(M^{(N)}, d\mu^{(1)\otimes N})$ that are absolutely continuous with respect to $\tau^{\otimes N}$. We will denote their space by X_N.

Lemma 3.2. *For $\beta\gamma \in (-\infty, 8)$ the functional $\mathcal{F}_\beta^{(N)}$ has a unique maximum and it is achieved by the measure $\mu^{(N)}$. That is*

$$\mathcal{F}^{(N)}(\beta) := \sup_{\varrho^{(N)} \in X_N} \mathcal{F}_\beta^{(N)}(\varrho^{(N)}) = \mathcal{F}_\beta^{(N)}(\mu^{(N)}). \tag{13}$$

Moreover,

$$\mathcal{F}_\beta^{(N)}(\mu^{(N)}) = \ln\left(\frac{\mathcal{M}^{(N)}(\beta)}{(\mathcal{M}^{(1)})^N}\right). \tag{14}$$

Proof. First, notice that $\mathcal{F}_\beta^{(N)}(\mu^{(N)})$ is well defined for $\beta \in (-\infty, \frac{8}{\gamma})$ and an explicit computation gives the equation (14).
Now,

$$\begin{aligned} \mathcal{F}_\beta^{(N)}\left(\varrho^{(N)}\right) = \beta \int_{M^{(N)}} U^{(N)} \frac{d\varrho^{(N)}}{d\mu^{(1)\otimes N}} d\mu^{(1)\otimes N}(dx_1, ...dx_N) \\ - \int_{M^{(N)}} \ln\left(\frac{d\varrho^{(N)}}{d\mu^{(1)\otimes N}}\right) \frac{d\varrho^{(N)}}{d\mu^{(1)\otimes N}} d\mu^{(1)\otimes N}(dx_1, ...dx_N). \end{aligned} \tag{15}$$

But

$$\frac{d\varrho^{(N)}}{d\mu^{(1)\otimes N}} = \frac{(\mathcal{M}^{(1)})^N}{\mathcal{M}^{(N)}(\beta)} e^{\beta U^{(N)}} \frac{d\varrho^{(N)}}{d\mu^{(N)}}.$$

Hence,

$$\begin{aligned} \mathcal{F}_\beta^{(N)}(\varrho^{(N)}) &= -\int_{M^{(N)}} \ln\left(\frac{d\varrho^{(N)}}{d\mu^{(N)}}\right) \varrho^{(N)}(dx_1, ..., dx_N) - \ln\left(\frac{(\mathcal{M}^{(1)})^N}{\mathcal{M}^{(N)}(\beta)}\right) \\ &= -\int_{M^{(N)}} \ln\left(\frac{d\varrho^{(N)}}{d\mu^{(N)}}\right) \varrho^{(N)}(dx_1, ..., dx_N) + \mathcal{F}_\beta^{(N)}(\mu^{(N)}), \end{aligned}$$

and using the fact that $x \ln x \geq x - 1$, with equality if and only if $x = 1$, we find that

$$\mathcal{F}_\beta^{(N)}(\varrho^{(N)}) - \mathcal{F}_\beta^{(N)}(\mu^{(N)}) \leq 0,$$

with equality holding if and only if $\varrho^{(N)} = \mu^{(N)}$. □

Next, we show a very important property of the sequence $\mathcal{F}^{(N)}(\beta)$.

Proposition 3.3. *Given $\beta < \frac{8}{\gamma}$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \mathcal{F}^{(N)}(\beta) =: f(\beta),$$

exists and is finite.

The proof of this proposition will follow from the next two lemmata.

Lemma 3.4. *The sequence $\frac{1}{N}\mathcal{F}^{(N)}(\beta)$ is bounded below and above independently of N.*

Proof: For the bound from below, we apply Jensen's inequality to $\mathcal{M}^{(N)}(\beta)$ with the concave function $\ln(\cdot)$. This leads to

$$\ln\left(\frac{\mathcal{M}^{(N)}(\beta)}{\left(\mathcal{M}^{(1)}\right)^N}\right) \geq \frac{N}{2}\beta\hat{\mu}^{(1)\otimes 2}(U(x,y)).$$

Hence,

$$\frac{1}{N}\mathcal{F}^{(N)}(\beta) \geq \frac{\beta}{2}\hat{\mu}^{(1)\otimes 2}(U(x,y)).$$

The bound from above, can be deduced the exact same way as in Proposition 3.1. $\square$

Lemma 3.5. *The sequence $N \to \mathcal{F}^{(N)}(\beta)$ is sub-additive. That is, if $N = N_1 + N_2$ then*

$$\mathcal{F}^{(N)}(\beta) \leq \mathcal{F}^{(N_1)}(\beta) + \mathcal{F}^{(N_2)}(\beta).$$

Proof:
We set $N = N_1 + N_2$, then we have

$$\begin{aligned}
\mathcal{F}_\beta^{(N)}(\mu^{(N)}) &= \mathcal{S}^{(N)}(\mu^{(N)}) + \frac{\beta}{2}N\hat{\mu}_2^{(N)}(U(x,y)) \\
&\leq \mathcal{S}^{(N_1)}(\mu_{N_1}^{(N)}) + \mathcal{S}^{(N_2)}(\mu_{N_2}^{(N)}) + \frac{\beta}{2}(N_1+N_2)\hat{\mu}_2^{(N)}(U(x,y)) \\
&\leq \mathcal{F}^{(N_1)}(\beta) + \mathcal{F}^{(N_2)}(\beta),
\end{aligned}$$

where in the first equation, we used the symmetry of U and $\mu^{(N)}$ and in the second inequality the sub-additivity of the entropy $\mathcal{S}$. $\square$

The boundedness from below and the sub-additivity provided by Lemma 3.4 and 3.5, ensure the result of Proposition 3.3.

3.2 Integrability

The objective now is to show compactness (in the weak sense) of the sequence $(\mu_n^{(N)})_N$. In order to do that, we need to show a uniform L^p-boundedness for the sequence in question. We claim that

Proposition 3.6. *There exists a constant $K(n, \beta\gamma)$ such that*

$$\mu_n^{(N)}(dx_1\cdots dx_n) \leq K(n,\gamma\beta)\exp\left(\frac{\beta}{N-1}\sum_{1\leq i<k\leq n}U(x_i,x_j)\right)\tau^{\otimes n}.$$

Proof: First, we write $(N-1)U^{(N)} = W^{(n)} + W^{(n,N-n)} + W^{(N-n)}$. Here, $W^{(n)}$ is the term involving $(x_1,\cdots,x_n)$, $W^{(N-n)}$ is the term involving $(x_{n+1},\cdots,x_N)$, and finally the term $W^{(n,N-n)}$ contains the mixed remaining variables. First notice that $\frac{1}{N-1}W^{(n)} \to 0$ as $N\to\infty$, hence $e^{\frac{\beta}{N-1}W^{(n)}} \in L^p(M^{(n)})$ for N big enough.

Next, we move to the term $W^{(n,N-n)} + W^{(N-n)}$. Indeed, we take $q = \frac{N-1}{2n}$ and $q' = \frac{N-1}{N-1-2n}$ and using Hölder's inequality we get

$$\left\| \exp\left(\frac{\beta}{N-1} \left[W^{(n,N-n)} + W^{(N-n)} \right] \right) \right\|_{L^1(M^{(N)})} \leq \left\| \exp\left(\frac{\beta}{N-1} W^{(n,n-N)} \right) \right\|_{L^q(M^{(N)})} \times \left\| \exp\left(\frac{\beta}{N-1} W^{(n-N)} \right) \right\|_{L^{q'}(M^{(N)})}.$$

The first integral can be bounded the same way as in Proposition 3.1 and the fact that

$$\begin{aligned}
\left\| \exp\left(\frac{\beta}{N-1} \sum_{k=1}^{n} U(x_k, x) \right) \right\|_{L^q(M)}^{N-n} &= \left\| \exp\left(\frac{\beta}{N-1} (-\gamma \sum_{k=1}^{n} \ln |x_k x^{-1}| \chi_{B_1(x_k)} + \tilde{H}(x)) \right) \right\|_{L^q(M)}^{N-n} \\
&\leq C^{\frac{N-n}{N-1}} \left\| \frac{1}{|x|^{\frac{n\gamma\beta}{N-1}}} \right\|_{L^q(B_1(0))}^{N-n} \\
&\leq C(n) \left\| \frac{1}{|x|^{\frac{\gamma\beta}{2}}} \right\|_{L^1(B_1(0))}^{\frac{2n(N-n)}{N-1}}. \qquad (16)
\end{aligned}$$

Next we deal with the second term, namely $\| \exp(\frac{\beta}{N} W^{(n-N)}) \|_{L^{q'}}$, where $q' = \frac{N-1}{N-2n-1}$. This can be written as:

$$\left\| \exp\left(\frac{\beta}{N-1} W^{(n-N)} \right) \right\|_{L^{q'}(M^{(N)})} = \mathcal{M}^{(N-n)} \left(\beta \frac{N-n-1}{N-2n-1} \right)^{1-\frac{2n}{N-1}}.$$

Notice that since $\lim_{N\to\infty} \frac{1}{N} \mathcal{F}^{(N)}(\beta)$ exists, we have that $\mathcal{M}^{(N-n)} \left(\beta \frac{N-n-1}{N-2n-1} \right)^{-\frac{2n}{N-1}}$ is uniformly bounded. Hence, it remains to bound $\mathcal{M}^{(N-n)} \left(\beta \frac{N-n-1}{N-2n-1} \right)$. Using Jensen's inequality with respect to the measure $d\mu^{(1)\otimes n}$, we have

$$\begin{aligned}
\mathcal{M}^{(N)}(\beta) &\geq \left(\mathcal{M}^{(1)} \right)^n \exp\left(\frac{n(2N-n-1)}{N-1} \beta \mu^{(1)\otimes 2}(U(x,y)) \right) \mathcal{M}^{(N-n)} \left(\frac{N-n-1}{N-1} \beta \right) \\
&\geq C(n,\beta) \mathcal{M}^{(N-n)} \left(\frac{N-n-1}{N-1} \beta \right).
\end{aligned}$$

We now consider the density $\rho^{(N-n)}$ defined by

$$\rho^{(N-n)} = \frac{\exp\left(\beta \frac{1}{N-2n-1} W^{(N-n)} \right)}{\mathcal{M}^{(N-n)} \left(\frac{N-n-1}{N-2n-1} \beta \right)}.$$

We will write $\langle X \rangle_N$ the average of X with respect to the density $\rho^{(N-n)}$ and the measure $\tau^{\otimes(N-n)}$. Therefore, we have

$$\begin{aligned}\frac{\mathcal{M}^{(N-n)}\left(\frac{N-n-1}{N-1}\beta\right)}{\mathcal{M}^{(N-n)}\left(\frac{N-n-1}{N-2n-1}\beta\right)} &= \left\langle \exp\left(-\frac{2n}{(N-1)(N-2n-1)}\beta W^{(N-n)}\right)\right\rangle_N \\ &\geq \exp\left(-\frac{2n}{(N-1)(N-2n-1)}\left\langle \beta W^{(N-n)}\right\rangle_N\right) \\ &= \exp\left(-2n\beta\partial_\beta\left(\frac{1}{N-1}\mathcal{F}^{(N-n)}\left(\frac{N-n}{N-2n-1}\beta\right)\right)\right).\end{aligned}$$

But recall that since $\beta \mapsto \frac{1}{N}\mathcal{F}^{(N)}(\beta)$ is convex (this is easily verified by taking two derivatives), the function $\beta \mapsto f(\beta)$ is also convex. In particular, its derivative exists almost everywhere and it is non-decreasing. So, for $\beta_0 \in (\beta, \frac{4}{\gamma})$, we have that

$$\frac{\mathcal{M}^{(N-n)}\left(\frac{N-n-1}{N-1}\beta\right)}{\mathcal{M}^{(N-n)}\left(\frac{N-n-1}{N-2n-1}\beta\right)} \geq C\exp\left(-2n\beta\partial_\beta^+(f(\beta_0))\right),$$

where $\partial_\beta^+ f(\beta_0) = \lim_{\beta\to\beta_0^+}\partial_\beta f(\beta)$. This last limit exists since $\partial_\beta f$ is non-decreasing and this finishes the proof. □

The previous proposition states that $\mu_n^{(N)}$ has a density with respect to $d\tau^n$ (or $d\mu^{(1)\otimes n}$), in $L^p(M^{(n)})$ for all $p \geq 1$. In particular the sequence $(\mu_n^{(N)})_N$ is weakly compact in the space $P(M^{(n)}) \cap L^p(M^{(n)})$. We want to characterize the limit points.

Proposition 3.7. *Let us consider a weakly convergent subsequence $\mu_n^{(a(N))}$ that converges weakly to a limit point, say $\mu(\beta) \in P_s(\Omega)$. Then the decomposition measure of $\mu(\beta)$ is concentrated at the maximizers of $\mathcal{F}_\beta^{(1)}$.*

Proof: Recall that

$$\begin{aligned}\mathcal{F}_\beta(\mu) &= \lim_{n\to\infty}\frac{1}{n}\mathcal{F}_\beta^{(n)}(\mu_n) \\ &= \int_{P_\mathcal{E}(M)} \mathcal{S}^{(1)}(\rho) + \frac{\beta}{2}\hat{\rho}^{\otimes 2}(U(x,y))\nu(d\rho|\mu). \qquad (17)\end{aligned}$$

In particular, if we set

$$A_\beta = \sup_{\rho\in P_\mathcal{E}(M)} \mathcal{S}^{(1)}(\rho) + \frac{\beta}{2}\hat{\rho}^{\otimes 2}(U(x,y)) = \sup_{\rho\in P_\mathcal{E}(M)} \mathcal{F}^{(1)}(\beta),$$

then one has

$$\sup_{\mu\in P^s(M)} \mathcal{F}_\beta(\mu) \leq A_\beta.$$

On the other hand, we have

$$\begin{aligned}\mathcal{F}^{(N)}(\beta) = \mathcal{F}_\beta^{(N)}\left(\mu^{(N)}\right) &\geq \mathcal{F}_\beta^{(N)}\left(\rho^{\otimes N}\right)\\ &\geq N\left(\mathcal{S}^{(1)}(\rho) + \frac{\beta}{2}\rho^{\otimes 2}(U(x,y))\right). \end{aligned} \tag{18}$$

Hence,

$$f(\beta) \geq A_\beta.$$

Next, we write $\alpha(N) = n\left\lfloor \frac{\alpha(N)}{n} \right\rfloor + m$ and using the sub-additivity of the entropy $\mathcal{S}$, we have

$$\begin{aligned}\mathcal{S}^{(\alpha(N))}(\mu^{(\alpha(N))}) &\leq \left\lfloor \frac{\alpha(N)}{n} \right\rfloor \mathcal{S}^{(n)}(\mu_n^{(\alpha(N))}) + \mathcal{S}^{(m)}(\mu_m^{(\alpha(N))})\\ &\leq \left\lfloor \frac{\alpha(N)}{n} \right\rfloor \mathcal{S}^{(n)}(\mu_n^{(\alpha(N))}).\end{aligned}$$

Using the upper-semicontinuity of the entropy, we have

$$\limsup_{N\to\infty} \mathcal{S}^{(n)}(\mu_n^{\alpha(N)}) \leq \mathcal{S}^{(n)}(\mu_n(\beta)).$$

Hence,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{\alpha(N)}\mathcal{S}^{(\alpha(N))}(\mu^{\alpha(N)}) &\leq \limsup \frac{1}{\alpha(N)}\left\lfloor \frac{\alpha(N)}{n} \right\rfloor \mathcal{S}^{(n)}(\mu_n^{(\alpha(N))})\\ &\leq \frac{1}{n}\mathcal{S}^{(n)}(\mu_n(\beta)).\end{aligned}$$

Therefore, if we let $n \to \infty$, we have

$$\limsup_{N\to\infty} \frac{1}{\alpha(N)}\mathcal{S}^{(\alpha(N))}(\mu^{\alpha(N)}) \leq \mathcal{S}(\mu(\beta)).$$

In particular

$$\begin{aligned}f(\beta) &= \limsup \frac{1}{\alpha(N)}\mathcal{F}_\beta^{(\alpha(N))}(\mu^{(\alpha(N))})\\ &\leq \mathcal{F}_\beta(\mu(\beta))\\ &\leq \sup_{\mu\in P^s(\Omega)} \mathcal{F}_\beta(\mu).\end{aligned}$$

Therefore, $A_\beta = f(\beta) = \mathcal{F}_\beta(\mu(\beta))$.

Thus the limiting points concentrate at the maximizers of A_β. Hence, $A_\beta = \max_{\rho\in P_\mathcal{E}(M)} \mathcal{F}_\beta^{(1)}(\rho)$.
In fact, one can see that the decomposition measure is actually concentrated on measures with density that is in $L^p(M)$ for all $p > 1$. $\square$

Next, to finish the proof of Theorem 2.5 we notice that as a consequence of Proposition 3.7, the maximization problem

$$f(\beta) = \sup\{\mathcal{F}_\beta^{(1)}; \rho \in P(M) \cap L^1 Log(L)(M)\}$$

has a solution and thus the solution satisfies the Euler-Lagrange equation

$$\rho_\beta(x) = \frac{Qe^{\beta \int_M U(x,y)\rho_\beta(y)\ dy}}{\int_M Qe^{\beta \int_M U(x,y)\rho_\beta(y)\ dy}\ dx}.$$

The fact that $\rho_\beta \in L^p(M)$ follows from the regularity result of the density of the sequence $\mu_n^{(N)}$.

3.3 Proof of the Main Result

We start by the case $Q > 0$. Using Theorem 2.5, we take $u = \frac{\beta}{2}\int_M U(x,y)\rho_\beta(y)dy + c$, where c is a constant to be determined later. Then we have that

$$\begin{aligned}\overline{P}'u(x) &= \frac{\beta}{2}\int_M \Big(\Gamma(x,y) - \frac{\overline{Q}'}{\int_M \overline{Q}'\ dv}\Big)\rho_\beta(y)\ dy \\ &= \frac{\beta}{2}\Big[\rho_\beta^T - \frac{\overline{Q}'}{\int_M \overline{Q}'\ dv}\Big],\end{aligned}$$

where $\rho_\beta = \rho_\beta^T + \rho_\beta^\perp$ and $\rho_\beta^T = \Gamma(\rho_\beta)$. Thus

$$\overline{P}'u + \frac{\beta}{2}\frac{\overline{Q}'}{\int_M \overline{Q}'dx} = \frac{\beta}{2\lambda}Qe^{2u-2c} - \frac{\beta}{2}\rho_\beta^\perp,$$

where $\lambda = \int_M Qe^{\beta \int_M U(x,y)\rho_\beta(y)\ dy}\ dx$. Since $0 < \int_M \overline{Q}'dx < 16\pi^2$, the number $\beta = 2\int_M \overline{Q}'dx$ satisfies $\beta\gamma \in [0,8)$. Moreover, one can pick $e^{-2c} = \frac{2\lambda}{\beta}$, to obtain a solution of

$$\overline{P}'u + \overline{Q}' = Qe^{2u} \mod \mathcal{P}^\perp.$$

On the other hand, if $Q < 0$, one can use Theorem 2.5 with $-Q$ instead of Q and $\beta < 0$ and the rest of the procedure follows as in the previous case.

4 Case of the Heisenberg Group

In this section we will extend the previous result to the non-compact case of the Heisenberg group. Notice that the estimates in the previous section rely on the compactness of the manifold M, so we need to adapt them to our new setting. We will be following the procedure developed in [11] and [28] for the Euclidean case. From now on we fix a "biharmonic" and pluriharmonic function K. That is K satisfies

$$(\Delta_b)^2 K = 0 \quad \text{and } T^2K = 0.$$

One such function would be $K(x,y,t) = -(x^2+y^2)$, but one could think of more complicated functions. We also consider the following two assumptions on K and $Q > 0$:

a) For all $0 < q < 4$, we have $\int_{B_1(x)} \frac{Q(y)e^{2K(y)}}{|xy^{-1}|^q}\, dy \to 0$ as $x \to \infty$, where $B_1(x)$ is the Heisenberg unit ball centered at x.

b) There exists $s \geq 0$ such that $\int_{\mathbb{H}} Q(x)e^{2K(x)}|x|^s dx < \infty$.

These assumptions will guarantee that the mass does not escape to infinity. We define $s^* := \sup\{s \geq 0; \int_{\mathbb{H}} Q(x)e^{2K(x)}|x|^s dx < \infty\}$ and $\beta^* = -2s^*$.

An explicit computation done in [32] shows that the Green's function of the operator P' or $\overline{P}'$ has the explicit form $G(x,y) = -\frac{1}{4\pi^2}\ln(|xy^{-1}|)$ and

$$\overline{P}' G(\cdot, y) = ReS(\cdot, y),$$

where S is the Szegő kernel. Therefore, we will take $U(x,y) = G(x,y)$. For the sake of notation, we will remove the factor $\frac{1}{4\pi^2}$ in the definition of U. The measure τ defined in (10) will be replaced by

$$\tau(dx) = e^{2K(x)}Q(x)dx.$$

Notice that from assumption (b), we have that the mass $\mathcal{M}^{(1)}$ of τ is finite and hence the probability measure $\mu^{(1)}$ is still well defined. The Hamiltonian $U^{(N)}$ then can be written as

$$U^{(N)}(x_1, \cdots, x_N) = -\ln(|R^{(N)}|^{\frac{1}{N-1}}),$$

where $R^{(N)} = \Pi_{1\leq i<j\leq N}|x_i x_j^{-1}|$. The definition of the entropy and the energy will remain unchanged. So as in Lemma 3.2, we have that $\mathcal{F}_\beta^{(N)}$ has a unique minimizer $\mu^{(N)}$ that can be written as

$$\begin{aligned} \mu^{(N)} &= \frac{1}{\mathcal{M}^{(N)}(\beta)}|R^{(N)}|^{-\frac{\beta}{N-1}} d\tau^{\otimes N} \\ &= \frac{1}{\mathcal{M}^{(N)}(\beta)} \exp\Big(\frac{-\beta}{N-1}\sum_{1\leq i<j\leq N}\ln(|x_i x_j^{-1}|)\Big) d\tau^{\otimes N}. \end{aligned} \tag{19}$$

For the well definedness of $\mu^{(N)}$ one needs to show that $\mathcal{M}^{(N)}(\beta)$ is finite.

Lemma 4.1. *The measure $\mu^{(N)}$ is absolutely continuous with respect to the measure $\tau^{\otimes N}$. Moreover, $\frac{d\mu^{(N)}}{d\tau^{\otimes N}} \in L^p(\mathbb{H}^N)$ for $p \in [1, \frac{8}{\beta})$, if $\beta > 0$ and $p \in [1, \frac{\beta^*}{\beta})$, if $\beta < 0$, for N large enough.*

Proof: We have for $p \geq 1$

$$\begin{aligned}\int_{\mathbb{H}^N} |R^{(N)}|^{-\frac{p\beta}{N-1}} \prod_{1\leq i\leq N} \tau(dx_i) &\leq \frac{1}{N}\int_{\mathbb{H}^N} \sum_{i=1}^{N} \prod_{1\leq j\leq N; j\neq i} |x_i x_j^{-1}|^{-\frac{pN\beta}{2(N-1)}} \prod_{1\leq i\leq N} \tau(dx_i) \\ &\leq \int_{\mathbb{H}^N} \prod_{2\leq j\leq N} |x_1 x_j^{-1}|^{-\frac{pN\beta}{2(N-1)}} \prod_{1\leq i\leq N} \tau(dx_i) \\ &\leq \int_{\mathbb{H}} \Big(\int_{\mathbb{H}} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy)\Big)^{N-1} \tau(dx) \\ &\leq \sup_{x\in\mathbb{H}} \Big(\int_{\mathbb{H}} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy)\Big)^{N-1} \mathcal{M}^{(1)},\end{aligned}$$

where we used the arithmetic-geometric inequality in the second inequality. Now, if $\beta > 0$, we have that

$$\begin{aligned}\int_{\mathbb{H}} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy) &= \int_{B_1(x)} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy) + \int_{\mathbb{H}\setminus B_1(x)} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy) \\ &\leq g(x) + \mathcal{M}^{(1)}.\end{aligned}$$

But using assumption (a), we have that $g(x) = \int_{B_1(x)} |xy^{-1}|^{-\frac{pN\beta}{2(N-1)}} \tau(dy)$ is in $L^\infty(\mathbb{H})$ as long as $\frac{Np}{N-1} < \frac{8}{\beta}$. The adaptation is clear for the case $\beta^* < \beta < 0$, using assumption (b). □

In order to get weak compactness of the measure $\mu^{(N)}$, we need a few Lemmata, including the uniform L^p boundedness of the marginals, as in Proposition 3.6.

Lemma 4.2. *Given $\beta \in (\beta^*, 8)$, there exists two constants c_1 and c_2 depending only on β such that*

$$c_1 \leq \beta \hat{\mu}_2^{(N)}(\ln |xy^{-1}|) \leq \beta \hat{\mu}^{(1)\otimes 2}(\ln |xy^{-1}|) \leq c_2.$$

Proof: For the last inequality, we use the fact that $|xy^{-1}| \leq c(|x| + |y|) \leq c(2 + |x|)(2 + |y|)$. Then from assumptions (a) and (b), we have that

$$\beta \hat{\mu}^{(1)\otimes 2}(\ln |xy^{-1}|) \leq c_2.$$

So we move to the second inequality. We define the function f_N by

$$f_N(\beta) = -\frac{2}{N} \ln(\hat{\mu}^{(1)\otimes N}(|R^{(N)}|^{-\frac{\beta}{N}}).$$

Using Jensen's inequality, we have that

$$f_N(\beta) \leq \frac{2\beta}{N(N-1)} \hat{\mu}^{(1)\otimes N}(\ln(|R^{(N)}|) \leq \beta \hat{\mu}^{(1)}(\ln |xy^{-1}|).$$

On the other hand, notice that

$$-2\mathcal{F}_\beta^{(N)}(\mu^{(N)}) = N f_N(\beta). \tag{20}$$

Therefore

$$f_N(\beta) = \frac{1}{N}(-2\mathcal{S}^{(N)}(\mu^{(N)}) - 2\beta\hat{\mu}^{(N)}(U^{(N)}),$$

and by the non-positivity of the entropy, we have

$$\beta\hat{\mu}^{(1)}(\ln|xy^{-1}|) \geq f_N(\beta) \geq -\frac{2}{N}\beta\hat{\mu}^{(N)}(U^{(N)}) = \beta\hat{\mu}_2^{(N)}(\ln|xy^{-1}|).$$

It remains to show the first inequality. Since $\beta \in (0,8)$, there exists $\varepsilon > 0$ such that $(1+\varepsilon)\beta \in (\beta^*, 8)$. By applying Jensen's inequality twice, we have that

$$\mathcal{M}^{(N)}((1+\varepsilon)\beta) \geq \mathcal{M}^{(N)}(\beta)\exp(-\frac{1}{2}N\varepsilon\beta\hat{\mu}_2^{(N)}(\ln|xy^{-1}|)).$$

Hence,

$$f_N(\beta(1+\varepsilon)) \leq f_N(\beta) + \varepsilon\beta\hat{\mu}_2^{(N)}(\ln|xy^{-1}|).$$

We now consider the function f_0 defined by

$$f_0(\beta) = -\ln\Big(\sup_{x\in\mathbb{H}}\int_{\mathbb{H}}|xy^{-1}|^{-\frac{\beta}{2}}\mu^{(1)}(dy)\Big).$$

Assumption (b) guaranties that $f_0(\beta)$ is well defined and finite and one can easily check that given $\beta \in (0,8)$, there exists $N_0 > 0$ such that for $N \geq N_0$, we have

$$f_N(\beta) \geq f_0((1+\varepsilon)\beta) + f_0(\beta). \tag{21}$$

Now from (20) and (21), we have that

$$f_0(\beta) + f_0((1+\varepsilon)\beta) \leq f_N(\beta) \leq -2A_\beta.$$

Thus, with ε even smaller if needed, we have

$$\beta\hat{\mu}_2^{(N)}(\ln|xy^{-1}|) \geq \frac{1}{\varepsilon}(f_N((1+\varepsilon)\beta) - f_N(\beta)) \geq (f_0((1+\varepsilon)^2\beta) + f_0((1+\varepsilon)\beta) - A_\beta) \geq c_1.$$

Lemma 4.3. *Given $\beta \in (\beta^*, 8)$, there exists $N_1 > 0$ such that for $N \geq N_1$, there exists a constant c_3 depending only on β such that*

$$\beta\hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln|xy^{-1}|) \leq c_3.$$

Proof: We start by the case $\beta \in (\beta^*, 0)$. In this case, we have

$$\beta\hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln|xy^{-1}|) \leq \hat{\mu}_1^{(N)}\Big(\int_{B_1(x)}\beta\ln|xy^{-1}|\mu_1(dy)\Big)$$
$$\hat{\mu}_1^{(N)}(c_3) = c_3.$$

For $\beta \in (0,8)$, we use the inequality $|xy^{-1}| \leq c(|x|+2)(|y|+2)$ to obtain

$$\hat{\mu}^{(1)} \otimes \hat{\mu}_1^{(N)}(\ln|xy^{-1}|) \leq \tilde{c} + \hat{\mu}^{(1)}(\ln(2+|x|)) + \hat{\mu}_1^{(N)}(\ln(2+|y|)).$$

Assumption (a) yields

$$\hat{\mu}^{(1)}(\ln(2+|x|)) \leq C_1.$$

Therefore, it remains to bound the second term. First, we have for $\beta' = (1 - \frac{1}{N-1})\beta$,

$$\begin{aligned}\hat{\mu}_1^{(N)}(\ln(2+|y|)) =& \frac{\mathcal{M}^{(N-1)}(\beta')}{\mathcal{M}^{(N)}(\beta)} \int_{H^{N-1}} \frac{|R^{(N-1)}|^{-\frac{\beta'}{N-2}}}{\mathcal{M}^{(N-1)}(\beta')} \times \\ & \Big(\int_{\mathbb{H}} \prod_{i=1}^{N-1} |x_i y^{-1}|^{-\frac{\beta}{N-1}} \ln(2+|y|)\tau(dy) \Big) \prod_{i=1}^{N-1} \tau(dx_i).\end{aligned}$$

We fix $s \in (0, s^*)$, where s^* is the sup of all $s > 0$ for which (b) holds. Using the inequality $e^X + Y\ln(Y) - Y \geq XY$, for

$$X = e^{s\ln(2+|y|)},$$

and

$$Y = \frac{1}{s} \int_{\mathbb{H}^{N-1}} \frac{|R^{(N-1)}|^{-\frac{\beta'}{N-2}}}{\mathcal{M}^{(N-1)}(\beta')} \prod_{i=1}^{N-1} |x_i y^{-1}|^{-\frac{\beta}{N-1}} \prod_{i=1}^{N-1} \tau(dx_i),$$

yields

$$\begin{aligned}C_N(\beta) &:= \hat{\mu}_1^{(N)}(\ln(2+|y|)) - \frac{\mathcal{M}^{(N-1)}(\beta')}{\mathcal{M}^{(N)}(\beta)} \int_{\mathbb{H}} \exp(s(2+|y|))\tau(dy) \\ &\leq -\frac{1}{s}(1 + \ln(s) + \beta' \hat{\mu}_2^{(N)}(\ln|x-y|)) \\ &\leq \tilde{c}_2(\beta),\end{aligned}$$

where the last inequality follows from Lemma 4.2. Clearly, from assumption (b), we have the finiteness of the integral $\int_{\mathbb{H}} \exp(s\ln(2+|y|))\tau(dy)$. Therefore, in order to finish the proof, it is enough to show the N-independent bound of the quotient $\frac{\mathcal{M}^{(N-1)}(\beta')}{\mathcal{M}^{(N)}(\beta)}$. This last bound will be more involved and needs a different approach from the previous estimates. It follows the same idea as in [11] and [28] but we will add it here for the sake of completion. We start by regularizing the potential $(x,y) \mapsto \ln|xy^{-1}|$ by defining the function

$$V_\varepsilon(x,y) = \frac{1}{|B_\varepsilon(0)|^2} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(y)} \ln|ab^{-1}| \; dadb.$$

By the Lebesgue differentiation theorem (which holds in the Heisenberg group $\mathbb{H}$. In fact, it holds for more general metric measure spaces with doubling measure [18, Section 3.4]), we have that $V_\varepsilon(x,y) \to \ln|xy^{-1}|$, for almost every $x, y \in \mathbb{H}$. Next, we define the quantity $\mathcal{M}_\varepsilon^{(N)}(\beta)$, by substituting $\ln(|xy^{-1}|)$ with $V_\varepsilon(x,y)$ in (19). We consider the Hilbert space $\mathcal{H}_\varepsilon$ obtained by the completion of the set of $C_0^\infty(\mathbb{H})$ functions with mean zero, under the dot product $\langle \cdot, \cdot \rangle_\varepsilon$ defined by

$$\langle f, g \rangle_\varepsilon = -\frac{\beta}{N-1} \int_{\mathbb{H}} \int_{\mathbb{H}} f(x) V_\varepsilon(x,y) f(y) \; dxdy.$$

We also consider the measures $\delta_y^\sharp \in \mathcal{H}_\varepsilon$ defined by

$$\delta_y^\sharp = \delta_y - \chi_{B_{r_0}},$$

where r_0 is picked so that $|B_{r_0}| = 1$. We introduce the function W_ε and the measure $\tilde{\tau}$ defined by

$$W_\varepsilon(x) = \int_{B_{r_0}} V_\varepsilon(x,y)\ dy - \frac{1}{2}\int_{B_{r_0}} V_\varepsilon(x,y)\ dy,$$

and

$$\tau = e^{\beta W_\varepsilon}\tilde{\tau}.$$

With this notation, an easy computation shows that

$$\begin{aligned}\mathcal{M}_\varepsilon^{(N)}(\beta) &= \int_{\mathbb{H}^N} \exp(-\frac{\beta}{N-1}\sum_{1\le i<j\le N} V_\varepsilon(x_i,x_j))\prod_{\ell=1}^N e^{\beta W_\varepsilon(x_\ell)}\tilde{\tau}(dx_\ell)\\ &= e^{-\frac{N\beta}{2(N-1)}V_\varepsilon(0,0)}\int_{\mathbb{H}^N}\exp(\frac{1}{2}\langle\delta_{(N)}^\sharp,\delta_{(N)}^\sharp\rangle_\varepsilon)\prod_{\ell=1}^N\tilde{\tau}(dx_\ell),\end{aligned}$$

where $\delta_{(N)}^\sharp = \sum_{i=1}^N \delta_{x_i}^\sharp$ and where we used the translation invariance of the measure in the Heisenberg group to write $V_\varepsilon(x_i,x_i) = V_\varepsilon(0,0)$. Now using Minlo's theorem for Gaussian functional integration (see [16]), we have the existence of a Gaussian average $Ave(\cdot)$ on the space of linear forms φ, on $\mathcal{H}_\varepsilon$, with $Ave(\varphi(\delta_x^\sharp) = 0$ and

$$Ave(\varphi(\delta_x^\sharp)\varphi(\delta_x^\sharp)) = \frac{\beta}{N-1}V_\varepsilon(x,y).$$

Therefore,

$$Ave(\exp(\varphi(\delta_{(N)}^\sharp))) = \exp(\frac{1}{2}\langle\delta_{(N)}^\sharp,\delta_{(N)}^\sharp\rangle_\varepsilon).$$

Hence,

$$\mathcal{M}_\varepsilon^{(N)}(\beta) = e^{-\frac{N\beta}{2(N-1)}V_\varepsilon(0,0)}Ave\Big(\big(\int_{\mathbb{H}}\exp(\varphi(\delta_x^\sharp))\tilde{\tau}(dx)\big)^N\Big).$$

Using Jensen's inequality, we have that

$$\mathcal{M}_\varepsilon^{(N)}(\beta) \ge \Big(\mathcal{M}_\varepsilon^{(N-1)}(\beta')\Big)^{\frac{N}{N-1}}.$$

Thus, after letting $\varepsilon \to 0$, one has

$$\frac{\mathcal{M}^{(N)}(\beta)}{\mathcal{M}^{(N-1)}(\beta')} \ge \Big(\mathcal{M}^{(N-1)}(\beta')\Big)^{\frac{1}{N-1}}.$$

But recall that $\liminf_{N\to\infty}\frac{1}{N}\mathcal{F}^{(N)}(\beta) \ge A_\beta$, therefore

$$\liminf_{N\to\infty}\frac{\mathcal{M}^{(N)}(\beta)}{\mathcal{M}^{(N-1)}(\beta')} \ge \mathcal{M}^{(1)}e^{-A_\beta},$$

which finishes the proof. □

Proposition 4.4 (Uniform Boundedness)**.** *Given $n \geq 1$ and $\beta \in (\beta^*, 8)$, there exists $N(n,\beta) \in \mathbb{N}$ and a constant $C(n,\beta)$ such that, for $N \geq N(n,\beta)$,*

$$\frac{d\mu_n^{(N)}}{d\tau^{\otimes n}} \leq C(n,\beta)|R^{(n)}|^{-\frac{\beta}{N-1}}.$$

Proof: First, we write

$$\frac{d\mu_n^{(N)}}{d\tau^{\otimes n}} = K(x_1,\cdots,x_n)\frac{|R^{(n)}|^{-\frac{\beta}{N-1}}}{\mathcal{M}^{(N)}(\beta)},$$

where

$$K(x_1,\cdots,x_n) = \int_{\mathbb{H}^{(N-n)}} \prod_{1\leq i\leq n<j\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{N-1}} \prod_{n\leq k<\ell\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{N-1}} \tau(dx_j).$$

Using Hölder's inequality, there exists $N(n,\beta)$ such that for $N > N(n,\beta)$ we have

$$\begin{aligned} K(x_1,\cdots,x_n) \leq &\Big(\int_{\mathbb{H}^{N-n}} \prod_{1\leq i\leq n<j\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{2n}} \tau(dx_j)\Big)^{-\frac{2n}{N-1}} \times \\ &\Big(\int_{\mathbb{H}^{N-n}} \prod_{n\leq i<j\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{N-1-2n}} \tau(dx_j)\Big)^{1-\frac{2n}{N-1}}. \end{aligned}$$

For the first term of the right hand side, we have

$$\begin{aligned} \int_{\mathbb{H}^{N-n}} \prod_{1\leq i\leq n<j\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{2n}} \tau(dx_j) &= \Big(\int_{\mathbb{H}} \prod_{i=1}^{n} |x_i x^{-1}|^{-\frac{\beta}{2n}} \tau(dx)\Big)^{N-n} \\ &\leq \Big(\frac{1}{n}\int_{\mathbb{H}} \sum_{i=1}^{n} |x_i x^{-1}|^{-\frac{\beta}{2}} \tau(dx)\Big)^{N-n} \\ &\leq \begin{cases} \Big(\sup_{y\in\mathbb{H}} \int_{\mathbb{H}} |yx^{-1}|^{-\frac{\beta}{2}} \tau(dx)\Big)^{N-n} & \text{if } \beta \geq 0 \\ \Big(C_n \int_{\mathbb{H}} (2+|x|)^{\frac{-\beta}{2}} \tau(dx)\Big)^{N-n} & \text{if } \beta < 0. \end{cases} \end{aligned}$$

Hence, the first term is bounded uniformly in N. For the second term, we first consider

$$\mathcal{A}_N = \Big(\int_{\mathbb{H}^{N-n}} \prod_{n\leq i<j\leq N} |x_i x_j^{-1}|^{-\frac{\beta}{N-1-2n}} \tau(dx_j)\Big)^{-\frac{2n}{N-1}} = \Big(\mathcal{M}^{(N-n)}(k(N)\beta)\Big)^{-\frac{2n}{N-1}},$$

where $k(N) = \frac{N-n-1}{N-2n-1}$. Notice here that when $N > N(n,\beta)$, $1 < k(N) < \frac{8}{\beta}$, if $\beta \in (0,8)$ and $1 < k(N) < \frac{\beta^*}{\beta}$, if $\beta \in (\beta^*, 0)$. Then clearly

$$\limsup_{N\to\infty} \mathcal{A}_N \leq \Big(\frac{e^{-\mathcal{A}_\alpha}}{\mathcal{M}^{(1)}}\Big).$$

Therefore, in order to finish the proof, one needs to bound $\frac{\mathcal{M}^{(N-n)}(k(N)\beta)}{\mathcal{M}^{(N)}(\beta)}$. Indeed, using Jensen's inequality

$$\begin{aligned}\frac{\mathcal{M}^{(N-n)}(k(N)\beta)}{\mathcal{M}^{(N)}(\beta)} \leq & \frac{1}{\left(\mathcal{M}^{(1)}\right)^n} \exp\Big(\frac{n(n-1)}{2(N-1)}\beta\hat{\mu}^{(1)\otimes 2}(\ln|xy^{-1}|)\Big)\times \\ & \exp\Big(n(1-\frac{n}{N-1})\beta\hat{\mu}^{(1)}\otimes\hat{\mu}_1^{(N-n),k}(\ln|xy^{-1}|)\Big)\times \\ & \exp\Big(-n(\frac{N-n-1}{N-1})k(N)\beta\hat{\mu}_2^{(N-n),k}(\ln|xy^{-1}|)\Big),\end{aligned}$$

where $\mu^{(N-n,k)}$ is defined the same way as $\mu^{(N)}$ with β switched with $K(N)\beta$. By Lemma 4.2, The first exponential term is then bounded uniformly with respect to N and since $k(N)\to 1$ as $N\to\infty$, using the upper bound in Lemma 4.2 and the upper bound in Lemma 4.3, we get the uniform boundedness of the the desired quantities. □

The last ingredient for the weak-compactness of the sequence $(\mu_n^{(N)})_{n\leq N}$ is its tightness, since we are working in a non-compact domain (see [2]). We recall that a sequence of probability measures $(p_k)_{k\geq 1}$ is said to be tight if for all $\varepsilon>0$, there exists a compact set K_ε such that

$$p_k(K_\varepsilon)\geq 1-\varepsilon, \text{ for all } k\geq 1.$$

So we show the following

Lemma 4.5. *The sequence $(\mu_n^{(N)})_{n\leq N}$ is tight.*

Proof: Using the symmetry of the measure $\mu_n^{(N)}$, it is enough to show tightness for the case $n=1$. Namely, we need to show that given $\varepsilon>0$, there exists $R(\varepsilon)$ such that

$$\mu_1^{(N)}(B_{R(\varepsilon)})\geq 1-\varepsilon.$$

Then we consider the map $h:\mathbb{H}\to\mathbb{R}$ defined by

$$h(y)=\int_{\mathbb{H}}\ln|yx^{-1}|\mu^{(1)}(dx)+C,$$

where C is a constant chosen so that h is positive. It is possible to choose such a constant since, by construction of $\mu^{(1)}$, h is continuous and $\lim_{y\to\infty}h(y)=+\infty$, uniformly in y. Therefore, from Lemma 4.3, given $\varepsilon>0$, there exists $R(\varepsilon)>0$, such that

$$\hat{\mu}_1^{(N)}(h(x))\frac{1}{\varepsilon}\leq\frac{C(\beta)}{\varepsilon}\leq\inf_{x\notin B_{R(\varepsilon)}}h(x).$$

Thus,

$$\begin{aligned}\hat{\mu}_1^{(N)}(h(x)) &\geq \hat{\mu}_1^{(N)}(h(x)\chi_{\mathbb{H}\setminus B_{R(\varepsilon)}}) \\ &\geq \frac{1}{\varepsilon}\hat{\mu}_1^{(N)}(h(x))\hat{\mu}_1^{(N)}(\chi_{\mathbb{H}\setminus B_{R(\varepsilon)}}) \\ &\geq \frac{1}{\varepsilon}\hat{\mu}_1^{(N)}(h(x))(1-\mu_1^{(N)}(B_{R(\varepsilon)})).\end{aligned}$$

The result follows after dividing by $\frac{1}{\varepsilon}\hat{\mu}_1^{(N)}(h(x))$. $\square$

Now given the weak compactness, the rest of the procedure of Section 3 can be carried out to prove the following

Theorem 4.6. *Given a function $Q > 0$ satisfying (a) and (b). Then, for any $\beta \in (\beta^*, 8)$, there exists $\rho_\beta \in L^p(\mathbb{H})$ for all $p \geq 1$, such that*

$$\rho_\beta(x) = \frac{Q(x)e^{K(x)-\beta\int_{\mathbb{H}}\ln|xy^{-1}|\rho_\beta(y)\ dy}}{\int_{\mathbb{H}} Q(x)e^{K(x)-\beta\int_{\mathbb{H}}\ln|xy^{-1}|\rho_\beta(y)\ dy}\ dx}.$$

Theorem 1.2 and Corollary 1.3 are a direct corollary of the previous theorem.

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