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Function spaces via fractional Poisson kernel on Carnot groups and applications

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FUNCTION SPACES VIA FRACTIONAL POISSON KERNEL ON CARNOT GROUPS AND APPLICATIONS

By

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Abstract. We provide a new characterization of homogeneous Besov and Sobolev spaces in Carnot groups using the fractional heat kernel and Poisson kernel. We apply our results to study commutators involving fractional powers of the sub-Laplacian.

1 Introduction

Besov and Sobolev spaces measure regularity of functions and are of central importance in the study of PDEs. There has been much work on these spaces and their characterization in different settings. Such alternative characterizations provide flexibility for applications. In this work, we give a new characterization of homogeneous Besov and Sobolev spaces in Carnot groups using the fractional heat kernel and Poisson kernel. We use this to study commutators involving fractional powers of the sub-Laplacian.

A Carnot group is a Lie group whose Lie algebra admits a stratification. This decomposes the Lie algebra as a direct sum of vector subspaces, the first of which is called the horizontal layer and generates the other subspaces via Lie brackets. Carnot groups have a rich geometric structure adapted to the horizontal layer, including translations, dilations, Carnot–Carathéodory (CC) distance, and a Haar measure [1, 17, 32, 38]. Carnot groups have been studied in contexts such as differential geometry [17], subelliptic differential equations [11, 25, 24], real and complex analysis [45, 40, 41]. For an introduction to Carnot groups from the point of view of this paper and for further examples, we refer to [11, 25, 45].

In the Euclidean case there have been characterizations of Besov and Sobolev spaces using multiple tools, mostly relying on the Fourier transform and Littlewood–Paley decompositions [12, 13, 14]. In Carnot groups there have been a few characterizations of such spaces, for instance using the heat kernel [43] and

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a spectral multiplier version of Besov spaces [28]. We also mention the use of a Littlewood–Paley decomposition in the study of the phase space in the Heisenberg group [5, 6]. This uses the Fourier transform in that setting. We also point out the extension of the characterization in [43] to the case of metric measure spaces with heat kernels satisfying a Gaussian bound, [16]. As we will see later on, the kernels that we will be using do not satisfy this bound.

The heat kernel and fractional heat kernel in Carnot groups have been studied for some time, e.g., see [23] and the references therein. The Poisson kernel in Carnot groups was introduced and studied in [26], but the fractional one is a recent discovery. It was first introduced and studied in [22] to exhibit a Harnack-type estimate for the fractional Laplacian. The method of construction follows the classical one introduced by Caffarelli and Silvestre in [15], but here using the spectral resolution of the sub-Laplacian. We also point out that there is also another construction for a different fractional Poisson kernel in the Heisenberg group for the conformal fractional sub-Laplacian in [27] and the construction relies mainly on the Fourier transform.

In this paper we start by defining a norm using the fractional heat kernel which ends up being equivalent to the classical homogeneous Besov norm as stated in Theorem 3.4. This procedure is close to the one of [43] and it does rely partially on the semigroup property of the fractional heat kernel. Next, we study different properties of the fractional Poisson kernel, allowing us, as stated in Theorem 4.6, to provide different equivalent norms to the classical homogeneous Besov spaces. The main challenge in this procedure is to bypass the use of the Fourier transform and still keep certain harmonic analysis properties of the different kernel we are considering. Also, in the same spirit, in Theorem 5.1, we provide a lower bound for the fractional Sobolev norms using a square-function-type quantity involving the convolution with the Poisson kernel and we finish in Proposition 5.5, by providing a characterization of the BMO norm.

Concerning applications of our results, the second characterization that we provide for Besov and Sobolev spaces appears to be well suited to the study of commutators involving fractional powers of the sub-Laplacian. We recall that, in [36], the first author provided a family of estimates for the commutator of the fractional sub-Laplacian using a more direct approach in estimating the singular kernel of the operator. In this work we provide an extended result, which generalizes many classical commutator estimates known in the Euclidean setting to the case of Carnot groups. For instance, in Theorems 6.5 and 6.6 we provide bilinear-type estimates for three-term commutators involving the fractional sub-Laplacian. In fact the first result (namely Theorem 6.5) provides L^p-type estimates and the second result deals with the borderline setting of bounding the Hardy norm. Also, in Theorem 6.7, we provide a proof of the Chanillo-type commutator estimates for the Carnot group setting. We follow closely the ideas provided in the Euclidean setting [35] to use the fractional Poisson kernel to simplify the expressions of the commutators. But we point out that in the Euclidean case, the estimates and characterizations of the different spaces was established separately in [12]. This is why, in our case, we first have to cross the difficulty of characterizing these spaces.

In general, commutator estimates are a fundamental tool in the study of the regularity of PDE, especially in the fractional setting. For instance, in Carnot groups, [36] gives applications to the study of the regularity and decay of solutions to the fractional CR-Yamabe problem, while [33] characterizes the asymptotic profile decomposition of Palais–Smale sequences for the same problem. In the Euclidean setting one has even more applications of commutator estimates [34, 44, 20, 21].

The structure of the paper is as follows.

In Section 2 we provide the necessary preliminaries on the structure of Carnot groups, the sub-Laplacian and the heat kernel.

In Section 3 we provide a characterization of Besov spaces using the fractional heat kernel. This is the kernel of the flow generated by the fractional power of the sub-Laplacian. The proof in this section follows the approach in [43], where an analogous characterization of Besov spaces was obtained using the standard (non-fractional) heat kernel and Poisson kernel.

In Section 4 and Section 5 we move to the characterization of Besov, Sobolev and BMO spaces using the fractional Poisson kernel. Here we generalize ideas in the Euclidean setting and avoid notions involving the Fourier transform because that is a tool that we cannot afford in Carnot groups in general.

In Section 6 we provide several applications of our results to estimates for commutators of fractional powers of the sub-Laplacian. Such estimates were established and studied in the Euclidean setting in [18, 20, 21, 35, 34, 44].

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2 Preliminaries

2.1 Carnot groups.

Definition 2.1. A connected and simply connected Lie group $(\mathbb{G}, \cdot)$ is a **Carnot group of step** k if its Lie algebra $\mathfrak{g}$ admits a **step** k **stratification**. This means that there exist non-trivial linear subspaces $V_1, \dots, V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \tag{2.1}$$

where $[V_1, V_i] = V_{i+1}$ for $1 \leq i < k$ and $[V_1, V_k] = \{0\}$. Here $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

Let $m_i = \dim(V_i)$ for $i = 1, \dots, k$. Define $h_0 = 0$ and $h_i = m_1 + \cdots + m_i$ for $i = 1, \dots, k$. We also use the notation $n := h_k$ and $m := m_1$. The **homogeneous dimension** of $\mathbb{G}$ is then defined by $Q := \sum_{i=1}^{k} i \dim(V_i)$.

Choose a family of left invariant vector fields $X = \{X_1, \dots, X_n\}$ adapted to the stratification of $\mathfrak{g}$, i.e., such that $X_{h_{j-1}+1}, \dots, X_{h_j}$ is a basis of V_j for each $j = 1, \dots, k$. This identifies $\mathfrak{g}$ with $\mathbb{R}^n$. Using exponential coordinates of the first kind we identify $\mathbb{G}$ with $\mathfrak{g}$ and hence with $\mathbb{R}^n$. With these coordinates, $X_i(0) = e_i$ for $i = 1, \dots, n$.

Definition 2.2. The sub-bundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_m$ plays a particularly important role in the theory. It is called the **horizontal bundle** $H\mathbb{G}$. The fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x), \dots, X_m(x)\}, \quad x \in \mathbb{G}.$$

We can endow each fiber of $H\mathbb{G}$ with a corresponding inner product $\langle \cdot, \cdot \rangle$ and with a norm $|\cdot|$ that make the basis $X_1(x), \dots, X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called **horizontal sections** and a vector of $H_x\mathbb{G}$ a **horizontal vector**. Each horizontal section is identified by its canonical coordinates with respect to this moving frame $X_1(x), \dots, X_m(x)$. This way, a horizontal section ϕ is identified with a function $\phi = (\phi_1, \dots, \phi_m) : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 2.3. For any $x \in \mathbb{G}$, the **left translation** $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined by $\tau_x z = xz$.

For any $\lambda > 0$, the **dilation** $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined as

$$\delta_\lambda(x) = (\lambda \xi_1, \dots, \lambda^k \xi_k), \tag{2.2}$$

where $x = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \equiv \mathbb{G}$.

The **Haar measure** of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is Lebesgue measurable, we write $|A|$ to denote its Lebesgue measure.

Let $|\cdot|:\mathbb{G}\to[0,\infty)$ denote a **symmetric homogeneous norm** on $\mathbb{G}$ [11], meaning:

- $|\cdot|$ is continuous,
- $|\delta_\lambda(x)|=\lambda|x|$ for every $\lambda>0$,
- $|x^{-1}|=|x|$.

Note that any two continuous homogeneous norms are equivalent, i.e., within constant multiplicative constant factors of each other. All the estimates we give are the same if the norm is changed up to changes in constants. We denote the ball centered at a point $x\in\mathbb{G}$ with radius $r>0$ by

$$B(x,r)=\{y\in\mathbb{G}:|y^{-1}x|<r\}.$$

We denote balls centered at the identity 0 by $B(r)=B(0,r)$.

Given two non-negative functions f and g, we shall write $f\lesssim g$ if there exists a constant C such that

$$f(x)\le Cg(x)$$

for all $x\in\mathbb{G}$. Similarly we shall write $f\approx g$ if $f\lesssim g$ and $g\lesssim f$.

Definition 2.4. Suppose $f:\mathbb{G}\to\mathbb{R}$ is a function for which X_jf exists for $1\le j\le m$. Then we define the **horizontal gradient** of f as the horizontal section whose coordinates are $(X_1f,\dots,X_mf)$:

$$\nabla_{\mathbb{G}}f:=\sum_{i=1}^{m}(X_if)X_i.$$

We denote by Δ_b the **positive sub-Laplacian** defined by

$$\Delta_bf:=\sum_{j=1}^{m}X_jX_jf$$

whenever f is a function such that X_jX_jf exists for $1\le j\le m$.

If $\Omega\subset\mathbb{G}$ is an open set, we define $C^\infty(\Omega)$ as in the classical case when Ω is a subset of $\mathbb{R}^n$. We will use the inequality

$$\|f(\cdot y)-f(\cdot)\|_{L^1}\lesssim|y|\,\|\nabla_{\mathbb{G}}f\|_{L^1}$$

for all $y\in\mathbb{G}$ and sufficiently smooth $f:\mathbb{G}\to\mathbb{R}$. This is a consequence of the Fundamental Theorem of Calculus.

2.2 Heat kernel. For every multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we denote

$$|\beta| = \beta_1 + \cdots + \beta_n$$

and

$$D^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n}$$

where

$$X_i^{\beta_i} = \underbrace{X_i X_i \cdots X_i}_{\beta_i-\text{times}}.$$

We also use the notation $(\partial/\partial x_i)^\beta$ or ∂^β to denote differentiation with respect to the standard basis of $\mathbb{R}^n$. The Schwartz space and space of distributions are defined as in the classical setting, which we now briefly recall.

Definition 2.5. We define the **Schwartz space** $\mathcal{S}(\mathbb{G})$ by identification of $\mathbb{G}$ with $\mathbb{R}^n$:

$$\begin{aligned}\mathcal{S}(\mathbb{G}) =\{\phi \in C^\infty(\mathbb{G})\colon P(\partial/\partial x_i)^\beta \phi \quad &\text{is bounded on } \mathbb{G}\\ &\text{for every polynomial } P \text{ and every multi-index } \beta\}.\end{aligned}$$

We equip $\mathcal{S}(\mathbb{G})$ with the following seminorms for multi-indices $\alpha, \beta \in \mathbb{N}^n$:

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{G}} |x^\alpha D^\beta \phi|.$$

The convolution of two functions $f, g : \mathbb{G} \to \mathbb{R}$ is defined whenever it makes sense by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(xy^{-1})g(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} g(y^{-1}x)f(y)\,\mathrm{d}y.$$

Definition 2.6. The continuous dual of $\mathcal{S}(\mathbb{G})$ with the family of seminorms $\|\cdot\|_{\alpha,\beta}$ is the space of **distributions** on $\mathbb{G}$, denoted $\mathcal{S}'(\mathbb{G})$.

The action of a distribution f on a Schwartz function ϕ is denoted $\langle f, \phi\rangle$. The **convolution** $f * \phi$ of f and ϕ is defined by

$$(f * \phi)(x) = \langle f, \tilde{\phi}\rangle$$

where $\tilde{\phi}(y) = \phi(y^{-1}x)$. If α is a multi-index, the **derivative** $\partial^\alpha f$ of a distribution f is defined by

$$\langle \partial^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial^\alpha \varphi\rangle \quad \varphi \in \mathcal{S}(\mathbb{G}).$$

Define the **parabolic version** of a Carnot group $\mathbb{G}$ by $\hat{\mathbb{G}} := \mathbb{R} \times \mathbb{G}$. This is a Carnot group where the group operation in the first coordinate is the usual addition and its homogeneous dimension is $Q + 2$. We define dilations on $\hat{\mathbb{G}}$ by

$$\hat{\delta}_\lambda(t, x) = (\lambda^2 t, \delta_\lambda(x)).$$

Definition 2.7. The **heat operator** is the operator $\mathcal{H}$ on $\hat{\mathbb{G}}$ defined by

$$\mathcal{H} := \partial_t + \Delta_b.$$

The heat operator is:

- translation invariant, i.e., for any $g \in \mathbb{G}$, $\mathcal{H}(u \circ \tau_g) = (\mathcal{H}(u)) \circ \tau_g$,
- homogeneous of degree 2, i.e., for any $\lambda > 0$, $\mathcal{H}(u \circ \hat{\delta}_\lambda) = \lambda^2 \mathcal{H} u$,
- hypoelliptic, i.e., if u is a distribution on $\hat{\mathbb{G}}$ such that $\mathcal{H}u$ is C^∞ in some open set Ω, then u must be C^∞ on Ω.

Definition 2.8. The heat operator $\mathcal{H}$ admits a fundamental solution h, usually called the **heat kernel** for $\mathbb{G}$.

Write $h_t(x) := h(t,x)$ and define for f locally integrable on $\mathbb{R}^n$:

$$H_t f(x) = (f * h_t)(x) = \int_{\mathbb{R}^n} h(t, y^{-1}x) f(y)\, \mathrm{d}y$$

whenever the integral exists. Then $\{H_t\}_{t>0}$ is called the **heat semigroup** for $\mathbb{G}$.

We now recall properties of h and H_t; see for example [11, 26] or [47, Section IV. 4]

Theorem 2.9. *The heat kernel h satisfies:*

1. $h \in C^\infty(\hat{\mathbb{G}} \setminus \{(0,0)\})$;
2. $h(\lambda^2 t, \delta_\lambda(x)) = \lambda^{-Q} h(t,x)$ *for every* $x \in \mathbb{G}$ *and* $t, \lambda > 0$;
3. $h(t,x) = 0$ *for every* $t < 0$ *and* $\int_{\mathbb{G}} h(t,x)\, \mathrm{d}x = 1$ *for every* $t > 0$;
4. $h(t,x) = h(t, x^{-1})$ *for every* $t > 0$ *and* $x \in \mathbb{G}$;
5. *there exists* $c \geq 1$ *(depending only on* $\mathbb{G}$*) such that for every* $x \in \mathbb{G}$ *and* $t > 0$

$$c^{-1} t^{-Q/2} \exp\Big(-\frac{|x|^2)}{c^{-1}t}\Big) \leq h(t,x) \leq c t^{-Q/2} \exp\Big(-\frac{|x|^2}{ct}\Big); \tag{2.3}$$

6. *for every non-negative integer* k *and* $\beta \in \mathbb{N}^n$*, there exists* $c = c(\beta, k) > 0$ *such that for every* $x \in \mathbb{G}$ *and* $t > 0$

$$\Big|\frac{\partial^k}{\partial t^k} D^\beta h(t,x)\Big| \leq c t^{-\frac{Q+j+2k}{2}} e^{-\frac{|x|^2}{t}}. \tag{2.4}$$

Further, for any $f \in L^1(\mathbb{R}^n)$ *and* $t > 0$*, we have* $H_t f \in C^\infty(\mathbb{R}^n)$ *and* $u(t,x) = H_t f(x)$ *solves* $\mathcal{H}u = 0$ *in* $(0,\infty) \times \mathbb{R}^n$*. Also* $u(t,x) \to f(x)$ *strongly in* $L^1(\mathbb{R}^n)$ *as* $t \to 0$*.*

We now recall that the fractional sub-Laplacian and its inverse can be expressed using the heat semigroup H_t as in [25].

Definition 2.10. We define the **fractional sub-Laplacian** by

$$(-\Delta_b)^\alpha f = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(1-\alpha)} \int_\varepsilon^\infty t^{-\alpha}(-\Delta_b)H_t f \,\mathrm{d}t \tag{2.5}$$

and

$$(-\Delta_b)^{-\alpha} f = \lim_{\eta\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^\eta t^{\alpha-1} H_t f \,\mathrm{d}t,$$

where $0 < \alpha < 1$ and $f \in L^2(\mathbb{G})$ is any function for which the relevant limit exists in L^2 norm.

We recall the following proposition [25, 26].

Proposition 2.11. *For $0 < \alpha < Q$ the integral*

$$R_\alpha(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} h(t,x)\,\mathrm{d}t, \quad x \in \mathbb{G},$$

converges absolutely and has the following properties:

- *R_α is a kernel of type α, i.e., is C^∞ away from 0 and homogeneous of degree $\alpha - Q$;*
- *$R_\alpha * R_\beta = R_{\alpha+\beta}$ for $\alpha, \beta > 0$ and $\alpha + \beta < Q$;*
- *R_2 is the fundamental solution of $-\Delta_b$, i.e., $(-\Delta_b)R_2 = \delta_0$;*
- *for $f \in L^p(\mathbb{G})$ and $1 < p < \infty$, we have $(-\Delta_b)^{-\alpha} f = f * R_{2\alpha}$.*

From Proposition 2.11 and Theorem 2.9 it follows that $R_\alpha(x) \approx |x|^{-Q+\alpha}$. Also the function $\rho(x) = (R_\alpha(x))^{\frac{1}{\alpha-Q}}$ defines a symmetric homogeneous norm which is smooth away from the origin and induces a quasi-distance equivalent to the left-invariant Carnot–Carathéodory distance.

In a similar way one can define the function $\tilde{R}_\alpha$, introduced in [22], for $\alpha < 0$ and $\alpha \notin \{0, -2, -4, \ldots\}$ by

$$\tilde{R}_\alpha(x) = \frac{\frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} h(t,x)\,\mathrm{d}t.$$

Again, $\tilde{R}_\alpha$ is homogeneous of degree $\alpha - Q$ and

$$\tilde{R}_\alpha(x) \approx |x|^{\alpha-Q}. \tag{2.6}$$

Using classical interpolation (or what is called λ-kernel estimates in [26]) one has, for $0 < \alpha < Q$,

$$\|R_\alpha u\|_{L^p} \lesssim \|u\|_{L^q} \tag{2.7}$$

for $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{Q}$ and $1 < q < Q$. Using $\tilde{R}_\alpha$ one can define another representation for the fractional sub-Laplacian. The following theorem is from [22, Theorem 3.11].

Theorem 2.12. *If* $u \in \mathcal{S}(\mathbb{G})$, *then for* $0 < \alpha < 1$:

$$
\begin{aligned}
(-\Delta_b)^\alpha u(x) &= P.V. \int_{\mathbb{G}} (u(y) - u(x)) \tilde{R}_{-2\alpha}(y^{-1}x)\, \mathrm{d}y \\
&= \lim_{\varepsilon \to 0^+} \int_{\mathbb{G} \setminus B(x,\varepsilon)} (u(y) - u(x)) \tilde{R}_{-2\alpha}(y^{-1}x)\, \mathrm{d}y.
\end{aligned}
$$

Moreover, using Balakrishnan's approach (see [7, 8]) and what is proved in [29, Lemma 8.5] (see also [23]) we also have the following formula

Theorem 2.13. *If* $u \in \mathcal{S}(\mathbb{G})$, *then for* $0 < \alpha < 1$:

$$
\begin{aligned}
(-\Delta_b)^\alpha u(x) &= -\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty t^{-\alpha-1}(H_t u(x) - u(x))\, dt \\
&= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1}(H_t u(x) - u(x))\, dt.
\end{aligned}
$$

The integral in the right-hand side must be interpreted as a Bochner integral in $L^2(\mathbb{G})$.

2.3 Spectral analysis in carnot groups.

We collect here some well-known results in Spectral Analysis which will be used later in the paper.

Since $-\Delta_b$ is self-adjoint with domain $\{f \in L^2(\mathbb{G}) : -\Delta_b f \in L^2(\mathbb{G})\}$, we can consider its spectral resolution $\int_0^\infty \lambda dE(\lambda)$. Then [25, (3.12)]

$$
(-\Delta_b)^\alpha = \int_0^\infty \lambda^\alpha\, \mathrm{d}E(\lambda),
$$

with domain

$$
W^{2\alpha,2}(\mathbb{G}) = \left\{ u \in L^2(\mathbb{G}) : \int_0^\infty \lambda^{2\alpha}\, \mathrm{d}\, \langle E(\lambda)u, u \rangle < \infty \right\}.
$$

Any bounded Borel measurable function m on $[0, \infty)$ defines an operator on $L^2(\mathbb{G})$ by

$$
m(-\Delta_b) = \int_0^\infty m(\lambda)\, \mathrm{d}E(\lambda).
$$

Let K_m denote the **convolution kernel** of the operator $m(-\Delta_b)$, namely K_m is a distribution on $\mathbb{G}$ satisfying

$$
m(-\Delta_b)u = u * K_m \quad \text{for } u \in \mathcal{S}(\mathbb{G}). \tag{2.8}
$$

If m is also compactly supported, then $K_m \in L^2(\mathbb{G})$ and there exists a regular Borel measure σ_m on $[0, \infty)$, whose support is the L^2 spectrum of $-\Delta_b$, such that [37, Theorem 3.10]:

$$
\int_{\mathbb{G}} |K_m(x)|^2\, \mathrm{d}x = \int_0^\infty |m(\lambda)|^2\, \mathrm{d}\sigma_m(\lambda).
$$

Remark 2.14. For any function f on $\mathbb{G}$ and any $\lambda > 0$, set $d_\lambda f(x) = f(\delta_\lambda(x))$. We claim:

$$d_\lambda^{-1}(-\Delta_b)^\alpha d_\lambda = \lambda^{2\alpha}(-\Delta_b)^\alpha. \tag{2.9}$$

Indeed, by Theorem 2.12 we get

$$\begin{aligned} d_\lambda^{-1}(-\Delta_b)^\alpha d_\lambda f(x) &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{G}\setminus B(\delta_{\frac{1}{\lambda}}x,\varepsilon)} (d_\lambda f(y) - d_\lambda f(\delta_{\frac{1}{\lambda}}x))\tilde{R}_{-2\alpha}(y^{-1}\delta_{\frac{1}{\lambda}}x)\,\mathrm{d}y \\ &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{G}\setminus B(\delta_{\frac{1}{\lambda}}x,\varepsilon)} (f(\delta_\lambda y) - f(x))\tilde{R}_{-2\alpha}(\delta_{\frac{1}{\lambda}}((\delta_\lambda y)^{-1}x))\,\mathrm{d}y \\ &= \lambda^{Q+2\alpha}\lim_{\varepsilon\to 0^+}\int_{\mathbb{G}\setminus B(\delta_{\frac{1}{\lambda}}x,\varepsilon)} (f(\delta_\lambda y) - f(x))\tilde{R}_{-2\alpha}((\delta_\lambda y)^{-1}x)\,\mathrm{d}y \end{aligned}$$

where in the last equality we used $\tilde{R}_{-2\alpha}(\delta_{\frac{1}{\lambda}}((\delta_\lambda y)^{-1}x) = \lambda^{Q+2\alpha}\tilde{R}_{-2\alpha}((\delta_\lambda y)^{-1}x)$. The conclusion follows by a change of variables.

Similar to above, one can check that for any $m \in L^\infty((0,\infty))$ we have

$$d_\lambda^{-1}m((-\Delta_b)^\alpha)d_\lambda = m(\lambda^{2\alpha}(-\Delta_b)^\alpha). \tag{2.10}$$

Remark 2.15. Given $m \in L^\infty((0,\infty))$ and $t > 0$, set

$$\hat{m}(\lambda) = m(\sqrt{\lambda}) \quad \text{and} \quad m^t(\lambda) = m(t\sqrt{\lambda}).$$

For any $f \in \mathcal{S}(\mathbb{G})$,

$$\begin{aligned} m^t((-\Delta_b))f &= m(t(-\Delta_b)^{\frac{1}{2}})f = d_t^{-1}m((-\Delta_b)^{\frac{1}{2}})d_t f = d_t^{-1}\hat{m}((-\Delta_b))d_t f \\ &= d_t^{-1}(d_t f * K_{\hat{m}}) \\ &= t^{-Q}(f * d_t^{-1}K_{\hat{m}}). \end{aligned}$$

Hence

$$K_{m^t}(x) = t^{-Q}K_{\hat{m}}(\delta_{\frac{1}{t}}(x)). \tag{2.11}$$

Now suppose that $\alpha > Q/2$ and fix $\eta \in C_0^\infty(0,\infty)$ not identically zero. If m satisfies

$$\sup_t \|\eta(\cdot)m(t\cdot)\|_{W^{\alpha,2}(\mathbb{R})} < \infty, \tag{2.12}$$

then by [19, Lemma 6], $K_m^t \in L^1(\mathbb{G})$ uniformly in $t \in (0,\infty)$. Here $W^{\alpha,2}(\mathbb{R})$ denotes the standard fractional Sobolev space of order α.

2.4 Semigroups. We recall a few properties of the semigroups generated by fractional powers of generators of strongly continuous semigroups. We refer to [48, Section 11, Chapter IX] for more information and all the missing proofs.

2.4.1 General semigroups It is well known that $(\Delta_b)^\alpha$ is the generator of a Markovian semigroup $\{e^{tA_\alpha}\}_{t>0}$ which is related to $\{e^{tA}\}_{t>0}$ by the subordination formula

$$e^{tA_\alpha}u = \int_0^\infty f_{t,\alpha}(s)e^{sA}u\,\mathrm{d}s \tag{2.13}$$

$$= \int_0^\infty f_{1,\alpha}(\tau)e^{\tau t^{1/\alpha}A}u\,\mathrm{d}\tau \tag{2.14}$$

where

$$f_{t,\alpha}(\lambda) = \begin{cases} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^\alpha}\,\mathrm{d}z & \text{if } \lambda \geq 0, \\ 0 & \text{if } \lambda < 0, \end{cases}$$

for $\sigma > 0, t > 0$ and $0 < \alpha < 1$. Thanks to [48, Proposition 2 of Section 11, Chapter IX], $f_{t,\alpha}(\lambda)$ is non-negative for $\lambda \geq 0$ and for $\lambda > 0$

$$\int_0^\infty f_{t,\alpha}(s)e^{-s\lambda}\,\mathrm{d}s = e^{-t\lambda^\alpha}. \tag{2.15}$$

Moreover,

$$f_{t,\alpha}(s) \leq \min\left\{\frac{1}{t^{1/\alpha}}, \frac{t}{s^{1+\alpha}}\right\} \tag{2.16}$$

and for $-\infty < \delta < \alpha$

$$\int_0^\infty f_{t,\alpha}(s)s^\delta\,\mathrm{d}s = \frac{\Gamma(1-\delta/\alpha)}{\Gamma(1-\delta)}t^{\delta/\alpha} \tag{2.17}$$

and if $\delta \geq \alpha$

$$\int_0^\infty f_{t,\alpha}(s)s^\delta\,\mathrm{d}s = +\infty. \tag{2.18}$$

2.4.2 Heat semigroup. Let us now examine the heat semigroup. In this case, $(-\Delta_b)^\alpha$ is defined in (2.5) and its domain is $W^{2\alpha,2}(\mathbb{G})$. We may use (2.13) and the equation $e^{t(-\Delta_b)}u = H_t u$ to write

$$e^{t(-\Delta_b)^\alpha}u(x) = \int_0^\infty f_{t,\alpha}(s)\left(\int_{\mathbb{G}} h(s, y^{-1}x)u(y)\,\mathrm{d}y\right)\mathrm{d}s \quad \text{for } u \in L^2(\mathbb{G}).$$

Hence, using Theorem 2.9 and (2.14), we have

$$\begin{aligned} e^{t(-\Delta_b)^\alpha}u(x) &= \int_0^\infty e^{\tau t^{1/\alpha}(-\Delta_b)}u(x)f_{1,\alpha}(\tau)\,\mathrm{d}\tau \\ &= \int_0^\infty \Big(\int_{\mathbb{G}} h(\tau t^{1/\alpha}, y^{-1}x)u(y)\,\mathrm{d}y\Big)f_{1,\alpha}(\tau)\,\mathrm{d}\tau \\ &= \int_{\mathbb{G}} \Big(\int_0^\infty h(\tau t^{1/\alpha}, y^{-1}x)f_{1,\alpha}(\tau)\,\mathrm{d}\tau\Big)u(y)\,\mathrm{d}y. \end{aligned}$$

Thus, the function

$$h_\alpha(t,y) = \int_0^\infty h(\tau t^{\frac{1}{\alpha}}, y)f_{1,\alpha}(\tau)\,\mathrm{d}\tau \tag{2.19}$$

is the integral kernel of the semigroup $e^{t(-\Delta_b)^\alpha}$, i.e.,

$$e^{t(-\Delta_b)^\alpha}u(x) = \int_{\mathbb{G}} h_\alpha(t, y^{-1}x)u(y)\,\mathrm{d}y \quad \text{for } u \in L^2(\mathbb{G}). \tag{2.20}$$

2.5 Besov spaces. We now recall the definition of the Besov space $B^s_{p,q}(\mathbb{G})$ [43].

Definition 2.16. Let $0 < s < 1$, $1 \le p \le \infty$ and $1 \le q \le \infty$.
The Besov space $B^s_{p,q}(\mathbb{G})$ is defined for $q < \infty$ by

$$B^s_{p,q}(\mathbb{G}) := \Big\{f \in L^p(\mathbb{G}) : \int_{\mathbb{G}} \Big(\frac{\|f(xy) - f(x)\|_{L^p}}{|y|^s}\Big)^q \frac{\mathrm{d}y}{|y|^Q} < \infty\Big\}.$$

The Besov space $B^s_{p,\infty}(\mathbb{G})$ is defined by

$$B^s_{p,\infty}(\mathbb{G}) := \Big\{f \in L^p(\mathbb{G}) : \sup_{y\neq 0} \frac{\|f(xy) - f(x)\|_{L^p}}{|y|^s} < \infty\Big\}.$$

We define the corresponding semi-norms by

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \Big(\int_{\mathbb{G}} (\frac{\|f(xy)-f(x)\|_{L^p}}{|y|^s})^q \frac{\mathrm{d}y}{|y|^Q}\Big)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{y\neq 0} \frac{\|f(xy)-f(x)\|_{L^p}}{|y|^s} & \text{if } q = \infty. \end{cases}$$

Note that $B^s_{p,q}$ can also be defined as the completion of $\mathcal{S}(\mathbb{G})$ with respect to $\|\cdot\|_{L^p} + \|\cdot\|_{\dot{B}^{p,q}_s}$.

3 Besov Spaces via Fractional Heat Kernel

In this section we will provide a characterization of Besov spaces using the fractional heat kernel. Throughout this section we fix $\alpha \in (0,1)$. We first collect some properties of the function h_α defined in (2.19) by

$$h_\alpha(t,y) = \int_0^\infty h(\tau t^{\frac{1}{\alpha}}, y)f_{1,\alpha}(\tau)\,\mathrm{d}\tau.$$

Proposition 3.1. *Given $\alpha \in (0,1)$, the function h_α has the following properties:*

(1) $h_\alpha \in C^\infty(\hat{\mathbb{G}} \setminus \{(0,0)\})$,
(2) $h_\alpha(\lambda^{2\alpha}t, \delta_\lambda(x)) = \lambda^{-Q} h_\alpha(t,x)$ *for every* $x \in \mathbb{G}$ *and* $t, \lambda > 0$,
(3) $h_\alpha(t,x) = 0$ *for every* $t < 0$ *and* $\int_{\mathbb{G}} h_\alpha(t,x)\,\mathrm{d}x = 1$ *for every* $t > 0$,
(4) $h_\alpha(t,x) = h_\alpha(t,x^{-1})$ *for every* $t > 0$ *and* $x \in \mathbb{G}$.

Proof. (1), (2) and (4) are trivial consequences of Theorem 2.9 (1), (2) and (4) respectively. Property (3) follows from Theorem 2.9 (3) by observing that for $t > 0$ we have

$$\int_{\mathbb{G}} h_\alpha(t,x)\,\mathrm{d}x = \int_{\mathbb{G}} \int_0^\infty h(\tau t^{\frac{1}{\alpha}}, x) f_{1,\alpha}(\tau)\,\mathrm{d}\tau\,\mathrm{d}x = \int_0^\infty f_{1,\alpha}(\tau)\,\mathrm{d}\tau = 1, \tag{3.1}$$

where the last equality is [48, Proposition 3, Chapter IX]. □

In what follows, given $f \in L^p(\mathbb{G})$ we will use the notation

$$\begin{aligned} u(t,x) := (h_\alpha * f)(t,x) &= \int_{\mathbb{G}} h_\alpha(t,y) f(y^{-1}x)\,\mathrm{d}y \\ &= \int_{\mathbb{G}} h_\alpha(t,xy^{-1}) f(y)\,\mathrm{d}y. \end{aligned} \tag{3.2}$$

Recall that n is the topological dimension of $\mathbb{G}$.

Proposition 3.2. *Let $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$. Then $h_\alpha(t,x)$ satisfies for $t > 0$ and $x \neq 0$:*

$$\left| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,x) \right| \lesssim \begin{cases} |x|^{-(Q+|\beta|+2\alpha k)} & \text{if } |x|^{2\alpha} \geq t, \\ t^{-\frac{Q+|\beta|+2\alpha k}{2\alpha}} & \text{if } |x|^{2\alpha} \leq t. \end{cases}$$

Further, for $1 \leq p \leq r \leq \infty$ and $\delta = Q(1/p - 1/r)$, we have for all $t > 0$

$$\left\| \frac{\partial^k}{\partial t^k} D^\beta u(t,x) \right\|_{L^r} \lesssim t^{-\frac{|\beta|+2\alpha k+\delta}{2\alpha}} \|f\|_{L^p}.$$

Proof. We start with the pointwise estimates for $k = 0$ and $\beta = 0$. By Proposition 3.1(2) we have

$$h_\alpha(r^{2\alpha}t, \delta_r(x)) = r^{-Q} h_\alpha(t,x).$$

If $|x|^{2\alpha} \geq t$, we have

$$h_\alpha(t,x) = |x|^{-Q} h_\alpha(t|x|^{-2\alpha}, \delta_{\frac{1}{|x|}}(x)) \leq |x|^{-Q} \sup_{\substack{|y|=1 \\ 0<t_0\leq 1}} h_\alpha(t_0, y).$$

Hence to complete, it suffices to prove that

$$\sup_{\substack{|y|=1\\0<t_0\le 1}} h_\alpha(t_0,y) < \infty.$$

Indeed, from the expression of h_α, Theorem 2.9, the boundedness of $h(t,y)$ on the set $|y| = 1$ and the fact that $f_{1,\alpha}(\tau)$ is continuous and integrable in τ [48, Proposition 3, Chapter IX] we have that

$$|h_\alpha(t_0,y)| \le \int_0^\infty |h(\tau t_0^{\frac{1}{\alpha}},y)| f_{1,\alpha}(\tau)\,\mathrm{d}\tau \lesssim \int_0^\infty f_{1,\alpha}(\tau)\,\mathrm{d}\tau < \infty.$$

On the other hand, if $|x|^{2\alpha} \le t$, then we have

$$h_\alpha(t,x) = h_\alpha(t,\delta_{t^{\frac{1}{2\alpha}}}\delta_{t^{-\frac{1}{2\alpha}}}(x)) \le t^{-\frac{Q}{2\alpha}} \sup_{0<|y|\le 1} h_\alpha(1,y).$$

The thesis follows if we prove that $\sup_{0<|y|\le 1} h_\alpha(1,y) < \infty$. Using again the expression of h_α and the fact that $h(\tau,y)$ is uniformly bounded if $\tau \ge 1$, we have

$$\begin{aligned}|h_\alpha(1,y)| &\le \int_0^1 |h(\tau,y)| f_{1,\alpha}(\tau)\,\mathrm{d}\tau + \int_1^\infty |h(\tau,y)| f_{1,\alpha}(\tau)\,\mathrm{d}\tau\\ &\lesssim \int_0^1 \tau^{-\frac{Q}{2}} e^{-\frac{|y|^2}{c\tau}} f_{1,\alpha}(\tau)\,\mathrm{d}\tau + 1.\end{aligned}$$

By (2.16), it holds that

$$\int_0^\infty e^{-\lambda a} f_{t,\alpha}(\lambda) d\lambda = e^{-ta^\alpha}.$$

Therefore, $f_{t,\alpha}$ is the density of an α-stable subordinator. Now from [10, eq. 14], we have for $t > 0$ and $\lambda > 0$,

$$f_{t,\alpha}(\lambda) \lesssim t\lambda^{-1-\alpha} e^{-t\lambda^{-\alpha}}.$$

Hence,

$$\int_0^1 \tau^{-\frac{Q}{2}} e^{-\frac{|y|^2}{c\tau}} f_{1,\alpha}(\tau)\,\mathrm{d}\tau \lesssim \int_0^1 \tau^{-(\frac{Q}{2}+1+\alpha)} e^{-\frac{1}{\tau^\alpha}}\,\mathrm{d}\tau < \infty$$

and we conclude as before. This provides us with the following estimate:

$$h_\alpha(t,y) \lesssim \max\left(t^{-\frac{Q}{2\alpha}}, \frac{1}{|y|^Q}\right).$$

The proof of the pointwise estimates of the derivatives follows from the formula

$$\frac{\partial^k}{\partial t^k} D^\beta h_\alpha(r^{2\alpha}t, \delta_r(x)) = r^{-(Q+|\beta|+2k\alpha)} \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,x)$$

and Theorem 2.9(6). Let us move to the L^p estimates. Using Young's inequalities, since $u(t,x) = (h_\alpha * f)(t,x)$, we have that

$$\left\| \frac{\partial^k}{\partial t^k} D^\beta u(t,x) \right\|_{L^p} \leq \left\| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,\cdot) \right\|_{L^1} \|f\|_{L^p}.$$

The conclusion follows observing that

$$\begin{aligned}
&\int_{\mathbb{G}} \left| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,x) \right| \mathrm{d}x \\
&\quad = \int_{|x|<t^{\frac{1}{2\alpha}}} \left| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,x) \right| \mathrm{d}x + \int_{|x|\geq t^{\frac{1}{2\alpha}}} \left| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,x) \right| \mathrm{d}x \\
&\quad \lesssim \int_{|x|<t^{\frac{1}{2\alpha}}} t^{-\frac{Q+|\beta|+2\alpha k}{2\alpha}} \mathrm{d}x + \int_{|x|\geq t^{\frac{1}{2\alpha}}} |x|^{-(Q+|\beta|+2\alpha k)} \mathrm{d}x \\
&\quad \lesssim t^{-\frac{|\beta|+2\alpha k}{2\alpha}} + C_Q \int_{t^{\frac{1}{2\alpha}}}^{\infty} r^{Q-1} r^{-(Q+|\beta|+2\alpha k)} dr \\
&\quad \lesssim t^{-\frac{|\beta|+2\alpha k}{2\alpha}}.
\end{aligned} \tag{3.3}$$

If q is chosen such that $1/r = 1/p + 1/q - 1 \geq 0$, then by Young's inequality

$$\begin{aligned}
\left\| \frac{\partial^k}{\partial t^k} D^\beta u(t,x) \right\|_{L^r} &\leq \left\| \frac{\partial^k}{\partial t^k} D^\beta h_\alpha(t,\cdot) \right\|_{L^q} \|f\|_{L^p} \\
&\lesssim t^{-\frac{|\beta|+2\alpha k+Q(1-1/q)}{2\alpha}} \|f\|_{L^p} \\
&\lesssim t^{-\frac{|\beta|+2\alpha k+\delta}{2\alpha}} \|f\|_{L^p}.
\end{aligned}$$

□

We recall the following useful lemma [39]:

Lemma 3.3. *Let (S_1, μ_1) and (S_2, μ_2) be σ-finite measure spaces. Fix a $\mu_1 \times \mu_2$-measurable function K for which there exists $C > 0$ such that*

1. *$|K(x,y)| \leq C$ for $\mu_1 \times \mu_2$ a.e. $(x,y) \in S_1 \times S_2$,*
2. *$\int_{S_1} |K(x,y)| d\mu_1(x) \leq C$ for μ_2 a.e. $y \in S_2$,*
3. *$\int_{S_2} |K(x,y)| d\mu_2(y) \leq C$ for μ_1 a.e. $x \in S_1$.*

Then the integral operator defined by $T(f) = \int_{S_2} K(x,y) f(y)\, \mathrm{d}\mu_2(y)$ is bounded from $L^p(S_2, \mu_2)$ to $L^p(S_1, \mu_1)$ for $1 \leq p \leq \infty$.

Let $0 < s < 1$, $1 \leq p \leq \infty$, $1 \leq q < \infty$ and recall the function u defined in (3.2). We consider the following semi-norm on Besov spaces:

$$\|f\|_{s,p,q} = \left(\int_0^\infty \left(t^{1-\frac{s}{2}} \left\| \frac{\partial u}{\partial t}(t,\cdot) \right\|_{L^p} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

We can now prove our characterization of Besov spaces using the fractional heat kernel. The following result will be crucial later.

Theorem 3.4. *Let* $0 < s < 1$, $0 < \alpha < 1$, $1 \leq p \leq \infty$ *and* $1 \leq q < \infty$. *Then for any* $f \in L^p(\mathbb{G})$ *we have*

$$\|f\|_{s,p,q} \approx \|f\|_{\dot{B}^{\alpha s}_{p,q}}.$$

Proof. We will show the equivalence of these two seminorms for $f \in \mathcal{S}(\mathbb{G})$ and the result will follow then by density. By Proposition 3.1(3), $\int_{\mathbb{G}} \frac{\partial h_\alpha}{\partial t}\, \mathrm{d}x = 0$, therefore by (3.2) and Proposition 3.1(4)

$$\frac{\partial u}{\partial t}(t,x) = \int_{\mathbb{G}} \frac{\partial h_\alpha}{\partial t}(t,y)(f(y^{-1}x) - f(x))\, \mathrm{d}y.$$

Denoting $\omega_p(y) = \|f(xy) - f(x)\|_{L^p}$ and using Minkowski's integral inequality, we get

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_{L^p} &= \left(\int_{\mathbb{G}} \left|\int_{\mathbb{G}} \frac{\partial h_\alpha}{\partial t}(t,y)(f(xy) - f(x))\, \mathrm{d}y\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \int_{\mathbb{G}} \left|\frac{\partial h_\alpha}{\partial t}(t,y)\right| \omega_p(y)\, \mathrm{d}y.\end{aligned}$$

Now using Proposition 3.2, we have

$$t^{1-\frac{s}{2}}\left\|\frac{\partial u}{\partial t}\right\|_{L^p} \lesssim \left(t^{1-\frac{s}{2}} \int_{|y|^{2\alpha} \geq t} |y|^{-(Q+2\alpha)} \omega_p(y)\, \mathrm{d}y + t^{-\frac{Q+s\alpha}{2\alpha}} \int_{|y|^{2\alpha} \leq t} \omega_p(y)\, \mathrm{d}y\right).$$

Hence,

$$\begin{aligned}&\left(\int_0^\infty \left(t^{1-\frac{s}{2}}\left\|\frac{\partial u}{\partial t}\right\|_{L^p}\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\quad\lesssim \left(\int_0^\infty \left(t^{1-\frac{s}{2}} \int_{|y|^{2\alpha} \geq t} |y|^{-(Q+2\alpha)} \omega_p(y)\, \mathrm{d}y\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\qquad+ \left(\int_0^\infty \left(t^{-\frac{Q+s\alpha}{2\alpha}} \int_{|y|^{2\alpha} \leq t} \omega_p(y)\, \mathrm{d}y\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\quad\lesssim \left(\int_0^\infty \left(\int_{\mathbb{G}} t^{1-\frac{s}{2}} |y|^{-2\alpha+\alpha s} \chi_{|y|^{2\alpha} \geq t}(y) |y|^{-\alpha s} \omega_p(y) |y|^{-Q}\, \mathrm{d}y\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\qquad+ \left(\int_0^\infty \left(\int_{\mathbb{G}} t^{-\frac{Q+\alpha s}{2\alpha}} |y|^{Q+\alpha s} \chi_{|y|^{2\alpha} \leq t}(y) |y|^{-\alpha s} \omega_p(y) |y|^{-Q}\, \mathrm{d}y\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &\quad= I_1 + I_2.\end{aligned} \tag{3.4}$$

For the integral I_1, we apply Lemma 3.3 with

- $(S_1, \mu_1) = ((0,\infty), \frac{\mathrm{d}t}{t})$,
- $(S_2, \mu_2) = (\mathbb{G}, \frac{\mathrm{d}y}{|y|^Q})$,
- $K = K_1(t,y) = t^{1-\frac{s}{2}} |y|^{\alpha s - 2\alpha} \chi_{|y|^{2\alpha} \geq t}$,

- p replaced by q,
- f replaced by $\tilde{f}(y) = |y|^{-\alpha s}\omega_p(y)$.

It is not hard to verify that the assumptions of the lemma are satisfied with $C = 2$. For instance, to verify Lemma 3.3(2) we need to show $\int_{S_1} |K(t, y)| d\mu_1(t) \leq C$. To see this we compute as follows:

$$\begin{aligned}\int_{S_1} |K(t, y)| d\mu_1(t) &= \int_0^\infty t^{1-\frac{s}{2}} |y|^{\alpha s - 2\alpha} \chi_{|y|^{2\alpha} \geq t} \frac{\mathrm{d}t}{t} = |y|^{\alpha s - 2\alpha} \int_0^{|y|^{2\alpha}} t^{-\frac{s}{2}} \,\mathrm{d}t \\ &= \frac{1}{1-\frac{s}{2}} |y|^{\alpha s - 2\alpha} |y|^{2\alpha(1-\frac{s}{2})} \\ &= \frac{1}{1-\frac{s}{2}}.\end{aligned}$$

To see $\tilde{f} \in L^q(S_2, \mu_2)$ we notice that for $|y| > 1$, $\omega_p(y) \leq 2\|f\|_{L^p}$ and hence

$$|\tilde{f}(y)|^q \lesssim \frac{1}{|y|^{q\alpha s}} \in L^1(|y| > 1, \mu_2).$$

Now for $|y| < 1$, since $f \in \mathcal{S}(\mathbb{G})$ we have $\omega_p(y) \lesssim |y| \|\nabla_{\mathbb{G}} f\|_{L^p}$. Thus

$$|\tilde{f}(y)|^q \lesssim |y|^{q(1-\alpha s)} \in L^1(|y| < 1, \mu_2).$$

Applying Lemma 3.3 with these parameters leads to the estimate:

$$I_1 \lesssim \left(\int_{\mathbb{G}} \frac{\omega_p(y)^q}{|y|^{Q+qs\alpha}} \,\mathrm{d}y \right)^{\frac{1}{q}}. \tag{3.5}$$

Similarly, in order to bound I_2, we use Lemma 3.3 with the same measure spaces, the same function $\tilde{f}$, and

$$K = K_2(t, y) = t^{-\frac{Q+\alpha s}{2s}} |y|^{Q+\alpha s} \chi_{|y|^{2\alpha} \leq t}.$$

Once again this yields

$$I_2 \lesssim \left(\int_{\mathbb{G}} \frac{\omega_p(y)^q}{|y|^{Q+qs\alpha}} \,\mathrm{d}y \right)^{\frac{1}{q}}. \tag{3.6}$$

Combining (3.5) and (3.6) we get

$$\begin{aligned}\|f\|_{s,p,q} &= \left(\int_0^\infty \left(t^{1-\frac{s}{2}} \left\| \frac{\partial u}{\partial t} \right\|_{L^p} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_{\mathbb{G}} \frac{\omega_p(y)^q}{|y|^{Q+qs\alpha}} \,\mathrm{d}y \right)^{\frac{1}{q}} = \|f\|_{\dot{B}^{\alpha s}_{p,q}}.\end{aligned} \tag{3.7}$$

On the other hand, we have

$$f(xy) - f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{t} \Big(- \frac{\partial}{\partial r} u(r, xy) + \frac{\partial}{\partial r} u(r, x) \Big) \, \mathrm{d}r + u(t, xy) - u(t, x).$$

Hence

$$\omega_p(y) \leq 2 \int_0^t \Big\| \frac{\partial}{\partial r} u(r, x) \Big\|_{L^p} \mathrm{d}r + \| u(t, xy) - u(t, x) \|_{L^p}.$$

But we know that

$$\| u(t, xy) - u(t, x) \|_{L^p} \lesssim |y| \, \| \nabla_{\mathbb{G}} u(t, x) \|_{L^p}.$$

By the semigroup property $u(t, \cdot) = h_\alpha(\frac{t}{2}, \cdot) * u(\frac{t}{2}, \cdot)$ [48] and Proposition 3.2, we get for any $i = 1, \ldots, m_1$:

$$\Big\| \frac{\partial}{\partial t} X_i u(t, \cdot) \Big\|_{L^p} \lesssim \Big\| X_i h_\alpha \Big(\frac{t}{2}, \cdot \Big) \Big\|_{L^1} \Big\| \frac{\partial}{\partial t} u \Big(\frac{t}{2}, \cdot \Big) \Big\|_{L^p} \lesssim t^{-\frac{1}{2\alpha}} \Big\| \frac{\partial}{\partial t} u \Big(\frac{t}{2}, \cdot \Big) \Big\|_{L^p}. \tag{3.8}$$

Since $\|X_i u(t, \cdot)\|_{L^\infty} \to 0$ as $t \to \infty$, we obtain

$$X_i u(t, x) = - \int_t^\infty \frac{\partial}{\partial r} X_i u(r, x) \, \mathrm{d}r.$$

Thus by (3.8)

$$\begin{aligned} \|X_i u(t, \cdot)\|_{L^p} &\lesssim \int_t^\infty r^{-\frac{1}{2\alpha}} \Big\| \frac{\partial}{\partial r} u \Big(\frac{r}{2}, \cdot \Big) \Big\|_{L^p} \mathrm{d}r \\ &\lesssim \int_{\frac{t}{2}}^\infty r^{-\frac{1}{2\alpha}} \Big\| \frac{\partial}{\partial r} u(r, \cdot) \Big\|_{L^p} \mathrm{d}r. \end{aligned}$$

Therefore,

$$\omega_p(y) \lesssim \int_0^t \Big\| \frac{\partial}{\partial r} u(r, x) \Big\|_{L^p} \mathrm{d}r + |y| \int_{t/2}^\infty r^{-\frac{1}{2\alpha}} \Big\| \frac{\partial}{\partial r} u(r, x) \Big\|_{L^p} \mathrm{d}r.$$

So, if one takes $t = |y|^{2\alpha}$, we have that

$$\begin{aligned} \|f\|_{\dot{B}^{\alpha s}_{p,q}} &= \Big(\int_{\mathbb{G}} (|y|^{-\alpha s} \omega_p(y))^q |y|^{-Q} \, \mathrm{d}y \Big)^{\frac{1}{q}} \\ &\lesssim \Big(\int_{\mathbb{G}} \Big(\int_0^{|y|^{2\alpha}} |y|^{-\alpha s} \Big\| \frac{\partial}{\partial t} u(t, x) \Big\|_{L^p} \mathrm{d}t \Big)^q |y|^{-Q} \, \mathrm{d}y \Big)^{\frac{1}{q}} \\ &\quad + \Big(\int_{\mathbb{G}} \Big(\int_{|y|^{2\alpha}/2}^\infty |y|^{1-\alpha s} t^{\frac{-1}{2\alpha}} \Big\| \frac{\partial}{\partial t} u(t, x) \Big\|_{L^p} \mathrm{d}t \Big)^q |y|^{-Q} \, \mathrm{d}y \Big)^{\frac{1}{q}} \\ &=: I_1 + I_2 \\ &\lesssim \Big(\int_0^\infty \Big(t^{1-\frac{s}{2}} \Big\| \frac{\partial}{\partial t} u(t, x) \Big\|_{L^p} \Big)^q \frac{\mathrm{d}t}{t} \Big)^{\frac{1}{q}} \\ &= \|f\|_{s,p,q}. \end{aligned}$$

Here we used Lemma 3.3 to get the last inequality. Indeed, we take

$$\tilde{f}(t) = t^{1-\frac{s}{2}} \left\| \frac{\partial}{\partial t} u(t,x) \right\|_{L^p},$$

$(S_1, \mu_1) = ((0,\infty), \frac{dt}{t})$ and $(S_2, \mu_2) = (\mathbb{G}, \frac{dy}{|y|^Q})$. For I_1, we consider the kernel

$$K_1(t,y) = |y|^{-\alpha s} t^{\frac{s}{2}} \chi_{t<|y|^{2\alpha}},$$

and for I_2 we consider the kernel

$$K_2(t,y) = t^{\frac{s}{2}-\frac{1}{2\alpha}} |y|^{1-\alpha s} \chi_{t>|y|^{\alpha}/2}.$$

This completes the proof. □

4 Besov spaces via fractional poisson kernel

In this section we characterize Besov spaces $B^s_{p,q}(\mathbb{G})$ using the fractional Poisson kernel. Recall [22, eq. 26] that for any $0 < \alpha < 1$ the fractional Poisson kernel can be written as

$$p_\alpha(t,x) = C_\alpha t^{2\alpha} \int_0^\infty r^{-(1+\alpha)} e^{-\frac{t^2}{4r}} h(r,x)\, dr,$$

where $C_\alpha = (4^\alpha \Gamma(\alpha))^{-1}$. From now on, we will let $\tilde{\delta}$ denote the Dirac mass.

Proposition 4.1. *Let $n \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$. Then*

$$|D^\beta p_\alpha(t,x)| \lesssim \begin{cases} |x|^{-(Q+|\beta|)} & \text{if } |x| \geq t, \\ t^{-(Q+|\beta|)} & \text{if } |x| \leq t. \end{cases}$$

Proof. Using Theorem 2.9(2) it is easy to see that

$$p_\alpha(\lambda t, \delta_\lambda(x)) = \lambda^{-Q} p_\alpha(t,x) \quad \text{for every } t > 0,\ x \in \mathbb{G} \text{ and } \lambda > 0.$$

Therefore, if $t \leq |x|$ we get

$$p_\alpha(t,x) = |x|^{-Q} p_\alpha\left(\frac{t}{|x|}, \delta_{\frac{1}{|x|}}(x)\right) \leq |x|^{-Q} \sup_{\substack{0<t_0\leq 1 \\ |y|=1}} p_\alpha(t_0,y).$$

Hence to estimate $|p_\alpha(t,x)|$ for $|x| \geq t$ it is enough to show that

$$\sup_{\substack{0<t_0\leq 1 \\ |y|=1}} p_\alpha(t_0,y) < \infty.$$

Indeed, from the expression of p_α and Theorem 2.9 (5) we have that

$$\begin{aligned}|p_\alpha(t_0,y)| &\lesssim t_0^{2\alpha}\int_0^\infty r^{-(1+\alpha)}e^{-\frac{t_0^2}{4r}}h(r,y)\,\mathrm{d}r\\ &\lesssim t_0^{2\alpha}\left(\int_0^1 r^{-(1+\alpha)}e^{-\frac{t_0^2}{4r}}\,\mathrm{d}r+\int_1^\infty r^{-(1+\alpha)}e^{-\frac{t_0^2}{4r}}r^{-\frac{Q}{2}}\,\mathrm{d}r\right)\\ &\lesssim I+II.\end{aligned}$$

The integral II can be easily bounded, indeed we have

$$II \lesssim \int_1^\infty r^{-(1+\alpha+\frac{Q}{2})}\,\mathrm{d}r<\infty.$$

For the integral I we have, by the change of variable $s=\frac{r}{t_0^2}$,

$$I\lesssim \int_0^{\frac{1}{t_0^2}} s^{-(1+\alpha)}e^{-\frac{1}{4s}}\,\mathrm{d}s\lesssim \int_0^\infty s^{-(1+\alpha)}e^{-\frac{1}{4s}}\,\mathrm{d}s<\infty.$$

It follows that if $t\le |x|$, then $p_\alpha(t,x)\lesssim |x|^{-Q}$. Similarly, for $|x|\le t$, we get

$$p_\alpha(t,x)=t^{-Q}p_\alpha(1,\delta_{\frac{1}{t}}(x))\le t^{-Q}\sup_{|y|\le 1}p_\alpha(1,y).$$

Therefore, the estimate follows, if one shows $\sup_{|y|\le 1}p_\alpha(1,y)<\infty$. Using [43, Theorem 4] we can estimate

$$\begin{aligned}p_\alpha(1,y) &\lesssim \int_0^{|y|^2} r^{-(1+\alpha)}e^{-\frac{1}{4r}}|y|^{-Q}\,\mathrm{d}r+\int_{|y|^2}^\infty r^{-(1+\alpha)}e^{-\frac{1}{4r}}r^{-\frac{Q}{2}}\,\mathrm{d}r\\ &= I+II.\end{aligned}$$

By the strong decay of the exponential term, we have that

$$II\lesssim \int_0^\infty r^{-(1+\alpha+\frac{Q}{2})}e^{-\frac{1}{4r}}\,\mathrm{d}r<\infty.$$

In order to bound I, one can assume that $|y|$ is small or else the bound is obvious. For any $N\in\mathbb{N}$, there exists $C=C(N)>0$ such that for $|y|$ sufficiently small we have

$$e^{-\frac{1}{4r}}\le Cr^N \text{ for all } r\in(0,|y|^2).$$

Hence if we pick $N>\frac{1}{2}Q+\alpha$, then

$$I\lesssim |y|^{-Q}\int_0^{|y|^2} r^{-(1+\alpha-N)}\,\mathrm{d}r\lesssim |y|^{-(Q+2(\alpha-N))}.$$

This gives the desired result, leading to $p_\alpha(t,x)\lesssim t^{-Q}$.

Finally, from the homogeneity of h we have

$$D^{\beta}p_{\alpha}(rt,\delta_r(x)) = r^{-(Q+|\beta|)}D^{\beta}p_{\alpha}(t,x).$$

The bounds then follow as in Proposition 3.2. □

If one wants to consider also derivatives in t, the bound becomes a bit more involved. We adopt the following notation: $f \lesssim_{k,n} g$ if $\partial_t^i D^{\beta} f \lesssim \partial_t^i D^{\beta} g$ for all $i \leq k$ and $|\beta| \leq n$.

Lemma 4.2. *For all $n, k \in \mathbb{N}$, $t \in (0,\infty)$ and $x \in \mathbb{G}$ we have*

$$p_{\alpha}(t,x) \lesssim_{k,n} \frac{t^{2\alpha}}{(t^2+|x|^2)^{\frac{Q+2\alpha}{2}}}.$$

Proof. We will write the proof for $\partial_t p_{\alpha}$; for higher derivatives the proof follows the same strategy. From the formula defining p_{α} we have that

$$\begin{aligned}\partial_t p_{\alpha}(t,x) &\lesssim t^{2\alpha-1}\int_0^{\infty} r^{-(1+\alpha)}e^{-\frac{t^2}{4r}}h(r,x)\,\mathrm{d}r + t^{2\alpha+1}\int_0^{\infty} r^{-(2+\alpha)}e^{-\frac{t^2}{4r}}h(r,x)\,\mathrm{d}r \\ &= I + II.\end{aligned} \tag{4.1}$$

Let us focus on I. Using the estimate $h(r,x) \lesssim r^{-\frac{Q}{2}}e^{-\frac{|x|^2}{cr}}$ and a substitution of the form $s = \frac{r}{|x|^2}$, we have

$$(t^2+|x|^2)^{\frac{Q+2\alpha}{2}} I \lesssim |x|^{2\alpha+Q} I \lesssim t^{2\alpha-1}\int_0^{\infty} s^{-(1+\alpha)}e^{-\frac{t^2}{4|x|^2 s}}s^{-\frac{Q}{2}}e^{-\frac{1}{cs}}ds,$$

for $t < |x|$. This last integral is uniformly bounded by $\int_0^{\infty} s^{-(1+\alpha+\frac{Q}{2})}e^{-\frac{1}{cs}}ds$, which is finite since the integrand at infinity behaves like $s^{-(1+\alpha+\frac{Q}{2})}$ and at zero it vanishes at infinite order. Therefore,

$$I \lesssim \frac{t^{2\alpha-1}}{(t^2+|x|^2)^{\frac{Q+2\alpha}{2}}}.$$

Using the same method, we see that the second integral satisfies

$$II \lesssim \frac{t^{2\alpha-1}}{(t^2+|x|^2)^{\frac{Q+2\alpha}{2}}}.$$

Let $t > |x|$. From the homogeneity of h, we have

$$\partial_t^k p_{\alpha}(rt,\delta_r x) = r^{-(Q+k)}\partial_t^k p_{\alpha}(t,x).$$

Therefore using exactly the same procedure as in the proof of Proposition 4.1, we get

$$\partial_t p_{\alpha}(t,x) \lesssim \frac{1}{t^{Q+1}}$$

and this proves our estimate. □

As in the previous section, given $f \in \mathcal{S}(\mathbb{G})$ we will denote

$$u(t,x) = (p_\alpha * f)(t,x) = \int_{\mathbb{G}} p_\alpha(t,y) f(y^{-1}x)\,\mathrm{d}y = \int_{\mathbb{G}} p_\alpha(t,xy^{-1}) f(y)\,\mathrm{d}y \tag{4.2}$$

For the reverse inequality in Theorem 4.6 we need a non-degeneracy condition, that is a Calderon-type formula. In the Euclidean setting one can make use of the Fourier transform, but in a more abstract setting such as ours we need a different tool. We took inspiration from [28] where a characterization of Besov spaces using the Littlewood–Paley approach is proved. Since we are dealing with continuous versions of the decomposition rather than the classical discrete one, we adopt the following notation. Given $\psi \in L^1(\mathbb{G})$, we denote by ψ_t the function

$$\psi_t(x) = t^{-Q}\psi(\delta_{\frac{1}{t}}(x)) \quad \text{for all } x \in \mathbb{G} \text{ and } t > 0.$$

Lemma 4.3. *There exists $\psi \in L^1(\mathbb{G})$ such that $\int_{\mathbb{G}} \psi = 0$ and*

$$\int_0^\infty t\psi_t * \partial_t p_\alpha(t) \frac{\mathrm{d}t}{t} = \tilde{\delta} \quad \text{in } \mathcal{S}'(\mathbb{G}). \tag{4.3}$$

Remark 4.4. Equation (4.3) is always to be interpreted to mean

$$\lim_{\varepsilon\to 0, A\to\infty} \int_\varepsilon^A t\psi_t * \partial_t p_\alpha(t) \frac{\mathrm{d}t}{t} = \tilde{\delta} \quad \text{in } \mathcal{S}'(\mathbb{G}). \tag{4.4}$$

More precisely, let $K_{A,\varepsilon} = \int_\varepsilon^A t\psi_t * \partial_t p_\alpha(t) \frac{\mathrm{d}t}{t} \in \mathcal{S}'(\mathbb{G})$, in the sense that

$$\langle K_{A,\varepsilon}, f\rangle_{\mathcal{S}'} = \int_{\mathbb{G}} K_{A,\varepsilon}(y) f(y)\,\mathrm{d}y.$$

The convergence $K_{A,\varepsilon} \to \tilde{\delta}$ in $\mathcal{S}'(\mathbb{G})$ means that $\langle K_{A,\varepsilon}, f\rangle_{\mathcal{S}'} \to \langle \tilde{\delta}, f\rangle_{\mathcal{S}'} = f(0)$ for all $f \in \mathcal{S}(\mathbb{G})$. Now, if we set $T_{A,\varepsilon} = K_{A,\varepsilon} * f$, then by definition of convolution of tempered distributions, we have for $\tilde{f}_x(y) = f(xy^{-1})$ that

$$T_{A,\varepsilon}(x) = \langle K_{A,\varepsilon}, \tilde{f}_x\rangle_{\mathcal{S}'},$$

for all $x \in \mathbb{G}$. Thus, by the convergence $K_{A,\varepsilon} \to \tilde{\delta}$ we have

$$T_{A,\varepsilon}(x) \to f(x),$$

for all $f \in \mathcal{S}(\mathbb{G})$ and all $x \in \mathbb{G}$.

Proof of Lemma 4.3:. Let us denote by $\{E_\lambda\}$ the spectral resolution of $-\Delta_b$ in $L^2(\mathbb{G})$. Let $\phi \in C^{2-\alpha}([0,\infty))$ be as in [22, Proposition 4.1]. By [22, Theorem 4.4], for any $u \in L^2(\mathbb{G})$ and $t > 0$ we have

$$v(\cdot,t) = \int_0^\infty \phi(\theta t^{2\alpha}\lambda^\alpha)\,\mathrm{d}E(\lambda)u = u * p_\alpha(\cdot,t)$$

where $\theta = (2\alpha)^{-2\alpha}$. Moreover, by [22, Proof of Theorem 4.4], the following formula holds:

$$\phi(\theta t^{2\alpha}\lambda^{\alpha}) = 2^{-(\alpha+1)}c_\alpha \lambda^{\alpha/2}t^{\alpha}\theta^{1/2}\int_0^\infty \tau^{-(\alpha+1)}e^{-\tau\sqrt{\lambda}t}e^{-\frac{\sqrt{\lambda}t}{4\tau}}\,\mathrm{d}\tau = H_\alpha(\sqrt{\lambda}t).$$

Here $c_\alpha > 0$ (for the precise expression see [22, Proposition 4.1]) and $H_\alpha : [0,\infty) \to \mathbb{R}$ denotes the continuous function defined as

$$H_\alpha(s) = 2^{-(\alpha+1)}c_\alpha\theta^{1/2}s^\alpha\int_0^\infty \tau^{-(\alpha+1)}e^{-\tau s}e^{-\frac{s}{4\tau}}\,\mathrm{d}\tau.$$

Therefore,

$$u * t\partial_t p_\alpha(\cdot,t) = \int_0^\infty \tilde{H}_\alpha(t\sqrt{\lambda})\,\mathrm{d}E(\lambda)u \tag{4.5}$$

where $\tilde{H}_\alpha(s) = sH'_\alpha(s)$.

Now we want to find a continuous function $G : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty \tilde{H}_\alpha(t\sqrt{\lambda})G(t\sqrt{\lambda})\frac{\mathrm{d}t}{t} = 1,$$

which is equivalent to

$$\int_0^\infty \tilde{H}_\alpha(s)G(s)\frac{\mathrm{d}s}{s} = 1.$$

Since $\tilde{H}_\alpha(s)$ is continuous and not equal to the zero function, there exists an interval $I = [\frac{a}{2}, 2b]$ with $a > 0$, where $|\tilde{H}_\alpha(s)| > 0$ for all $s \in I$. Let η be a smooth function supported in $[\frac{a}{2}, 2b]$, equals 1 on $[a,b]$ and such that $0 \le \eta(s) \le 1$ in I. For all $s \in I$ we define

$$G_1(s) = \eta(s)\frac{s}{\tilde{H}_\alpha(s)}.$$

Therefore,

$$\int_0^\infty \tilde{H}_\alpha(s)G_1(s)\frac{\mathrm{d}s}{s} = \int_{a/2}^{2b}\eta(s)\,\mathrm{d}s.$$

Hence it suffices to take

$$G(s) = \frac{G_1(s)}{\int_{a/2}^{2b}\eta(t)\,\mathrm{d}t}.$$

Define $\hat{G} : [0,\infty) \to \mathbb{R}$ by $\hat{G}(\lambda) = G(\sqrt{\lambda})$. Since G has compact support, then using the results of Section 2.3, there exists $K_{\hat{G}} \in L^2(\mathbb{G})$ such that for all $u \in \mathcal{S}(\mathbb{G})$

$$\hat{G}(-\Delta_b)u = u * K_{\hat{G}}.$$

For every $t > 0$ we define $\hat{G}^t(\lambda) = G(t\sqrt{\lambda})$, from (2.11) we get

$$K_{\hat{G}^t}(x) = t^{-Q}K_{\hat{G}}(\delta_{\frac{1}{t}}(x)).$$

To conclude the proof it suffices to take $\psi(x) = K_{\hat{G}}(x)$ which gives $\psi_t(x) = K_{\hat{G}^t}(x)$. Indeed, by (4.5), for all $u \in L^2(\mathbb{G})$

$$\int_\varepsilon^A u * t\partial_t p_\alpha(\cdot, t)\frac{\mathrm{d}t}{t} = \int_\varepsilon^A \int_0^\infty \tilde{H}_\alpha(t\sqrt{\lambda})\,\mathrm{d}E(\lambda)u\frac{\mathrm{d}t}{t}. \tag{4.6}$$

Taking $u = f * \psi_t$ we get

$$u = \hat{G}^t((-\Delta_b))f = \int_0^\infty G(t\sqrt{\lambda})\,\mathrm{d}E(\lambda)f.$$

Therefore, (4.6) implies

$$\begin{aligned}\int_\varepsilon^A t\psi_t * \partial_t p_\alpha(t) * f\frac{\mathrm{d}t}{t} &= \int_\varepsilon^A \int_0^\infty \tilde{H}_\alpha(t\sqrt{\lambda})G(t\sqrt{\lambda})\,\mathrm{d}E(\lambda)f\frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \int_\varepsilon^A \tilde{H}_\alpha(t\sqrt{\lambda})G(t\sqrt{\lambda})\frac{\mathrm{d}t}{t}\,\mathrm{d}E(\lambda)f.\end{aligned}$$

Since, $\int_\varepsilon^A \tilde{H}_\alpha(t\sqrt{\lambda})G(t\sqrt{\lambda})\frac{\mathrm{d}t}{t} \to 1$ as $A \to \infty$ and $\varepsilon \to 0$, for all $\lambda > 0$, we have that

$$\lim_{A\to\infty,\varepsilon\to 0}\int_\varepsilon^A t\psi_t * \partial_t p_\alpha(t) * f\frac{\mathrm{d}t}{t} = \int_0^\infty \mathrm{d}E(\lambda)f = f.$$

Here the first equality follows from the fact that, for the spectral measure $\mathrm{d}E$, one has

$$\int f(\lambda)g(\lambda)\,\mathrm{d}E(\lambda) = \int f(\lambda)\,\mathrm{d}E(\lambda)\int g(\lambda)\,\mathrm{d}E(\lambda)$$

for all bounded Borel functions f and g. The second equality follows from Fubini's theorem. One way to see this is to localize at given functions f and g, that is, in the form

$$\left\langle \int_a^b \int_0^\infty G(t,\lambda)\,\mathrm{d}E(\lambda)f, g\right\rangle_{L^2}\mathrm{d}t = \int_a^b \int_0^\infty G(t,\lambda)\,\mathrm{d}\mu_{f,g}(\lambda)\,\mathrm{d}t$$

and then use the classical Fubini theorem for the measures $\mathrm{d}\mu_{f,g}$ and $\mathrm{d}t$ since now the measures are real valued, not operator valued. The last convergence statement follows again from the fact that if f_n converges pointwise to f, then the corresponding operators converge in the weak $*$-topology, that is

$$\left\langle \int f_n(\lambda)\,\mathrm{d}E(\lambda)u, v\right\rangle_{L^2} \to \left\langle \int f(\lambda)\,\mathrm{d}E(\lambda)u, v\right\rangle_{L^2}$$

for all $u, v \in L^2$.

Finally, by [19, Theorem 1, Lemma 6], we have that $\|\psi\|_{L^1}$ is finite. Moreover, notice that $G(s) = s^2G_2(s)$, where

$$G_2(s) = \frac{\eta(s)}{s\tilde{H}_\alpha(s)\int_{a/2}^{2b}\eta(t)\,\mathrm{d}t}.$$

Notice that G_2 is well defined and smooth since it is supported away from 0. Setting $\hat{G}_2(\lambda) = G_2(\sqrt{\lambda})$, we have that

$$\hat{G}(-\Delta_b) = -\Delta_b \hat{G}_2(-\Delta_b).$$

In particular, we have that

$$K_{\hat{G}} = -\Delta_b K_{\hat{G}_2}.$$

Therefore,

$$\int_{\mathbb{G}} \psi(x)\,\mathrm{d}x = \int_{\mathbb{G}} -\Delta_b K_{\hat{G}_2}(x)\,\mathrm{d}x = 0. \qquad \square$$

Remark 4.5. Notice that a similar construction can be also done for $t^r(-\Delta_b)^{\frac{r}{2}} p_\alpha$ for $r \in [0, 2]$. Namely, there for $r \in [0, 1]$, there exists $\psi \in L^1(\mathbb{G})$, with $\int_{\mathbb{G}} \psi = 0$ and $\int_0^\infty t\psi_t * t^r(-\Delta_b)^r p_\alpha(t) \frac{dt}{t} = \tilde{\delta}$. Indeed the spectral multiplier corresponding to $t^r(-\Delta_b)^{\frac{r}{2}} p_\alpha$ can be written as $(\sqrt{\lambda} t)^r H(\sqrt{\lambda} t)$. Hence one needs to find G such that

$$\int_0^\infty s^r H(s) G(s) \frac{\mathrm{d}s}{s} = 1.$$

But then, using the same idea as before, we can pick

$$G_1(s) = \eta(s) \frac{s^{1-r}}{H(s)}.$$

Another important remark is that if ψ is chosen as in the proof of Lemma 4.3, then $-\Delta_b(\psi) \in L^1$. In particular, by real interpolation, we have

$$\|\nabla_{\mathbb{G}} \psi\|_{L^1} \lesssim \|\psi\|_{L^1}^{\frac{1}{2}} \|\psi\|_{S^1_2}^{\frac{1}{2}} \lesssim \|-\Delta_b \psi\|_{L^1} + \|\psi\|_{L^1}.$$

Hence $\nabla_{\mathbb{G}} \psi \in L^1(\mathbb{G})$.

Theorem 4.6. *Let $f \in \mathcal{S}(\mathbb{G})$ and u be as in* (4.2). *Then for $s \in (0, 1)$ we have*

$$\left(\int_0^\infty (t^{1-s} \|\nabla_{\mathbb{G}} u(t, x)\|_{L^p})^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \approx \|f\|_{\dot{B}^s_{p,q}}.$$

Also, for $s < 2\alpha < 2$,

$$\left(\int_0^\infty \left(t^{1-s} \left\| \frac{\partial}{\partial t} u(t, x) \right\|_{L^p} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \approx \|f\|_{\dot{B}^s_{p,q}}$$

and for $s \in (0, 2)$,

$$\left(\int_0^\infty (t^{2-s} \|(-\Delta_b) u(t, x)\|_{L^p})^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \approx \|f\|_{\dot{B}^s_{p,q}}.$$

Proof. We begin with the proof of the first part. In this case, one inequality is easy to prove but the opposite one needs another ingredient provided by Lemma 4.3. First, notice that the fact $h(t,x) = h(t,-x)$ implies

$$\int_{\mathbb{G}} X_i h(r,x)\,\mathrm{d}x = 0 \quad \text{for all } i = 1,\dots,m \text{ and all } r \in (0,\infty).$$

Hence using the explicit form of p_α yields

$$\int_{\mathbb{G}} \nabla_{\mathbb{G}} p_\alpha(t,x)\,\mathrm{d}x = 0.$$

Next we have

$$\nabla_{\mathbb{G}} u(t,x) = \int_{\mathbb{G}} \nabla_{\mathbb{G}} p_\alpha(t,y)(f(xy) - f(x))\,\mathrm{d}y.$$

Therefore,

$$\|\nabla_{\mathbb{G}} u\|_{L^p} \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}} p_\alpha(t,y)|\omega_p(y)\,\mathrm{d}y,$$

where $\omega_p(y) = \|f(xy) - f(x)\|_{L^p}$. Thus, by Proposition 4.1

$$t^{1-s}\|\nabla_{\mathbb{G}} u\|_{L^p} \lesssim t^{1-s}\int_{|y|\geq t} |y|^{-(Q+1)}\omega_p(y)\,\mathrm{d}y + \int_{|y|<t} t^{-(Q+s)}\omega_p(y)\,\mathrm{d}y.$$

Again here, we use the same trick as in (3.4). Indeed, for the first integral we take $K(t,y) = \chi_{|y|>t}\frac{t^{1-s}}{|y|^{1-s}}$ on the spaces $((0,+\infty), \frac{\mathrm{d}t}{t})$ and $(\mathbb{G}, \frac{\mathrm{d}y}{|y|^Q})$ and $f(y) = \frac{\omega_p(y)}{|y|^s}$. For the second integral, we take $K(t,y) = \chi_{|y|<t}t^{-(Q+s)}|y|^{Q+s}$ with the same measure spaces and function f. We then have

$$\left(\int_0^\infty (t^{1-s}\|\nabla_{\mathbb{G}} u(t,x)\|_{L^p})^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}}.$$

This proves the first inequality.

For the reverse inequality, we first use Remark 4.5 to see that

$$f(x) = \int_0^\infty (\psi_t * t(-\Delta_b)^{\frac{1}{2}} p_\alpha * f)(x)\frac{\mathrm{d}t}{t}.$$

Hence, using Young's inequality for convolutions in the first inequality,

$$\begin{aligned}
&\|f(xy) - f(x)\|_{L^p_x} \\
&\lesssim \int_0^\infty t\|\psi_t(xy) - \psi_t(x)\|_{L^1_x}\|(-\Delta_b)^{\frac{1}{2}} u\|_{L^p}\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \chi_{|y|\geq t} t\|\psi\|_{L^1}\|(-\Delta_b)^{\frac{1}{2}} u\|_{L^p}\frac{\mathrm{d}t}{t} \\
&\quad + \int_0^\infty \chi_{|y|\leq t}\|t\nabla_{\mathbb{G}}\psi_t\|_{L^1}|y|\|(-\Delta_b)^{\frac{1}{2}} u\|_{L^p}\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty \chi_{|y|\geq t} t\|\psi\|_{L^1}\|(-\Delta_b)^{\frac{1}{2}} u\|_{L^p}\frac{\mathrm{d}t}{t} + \int_0^\infty \chi_{|y|\leq t}\|\nabla_{\mathbb{G}}\psi\|_{L^1}|y|\|(-\Delta_b)^{\frac{1}{2}} u\|_{L^p}\frac{\mathrm{d}t}{t}.
\end{aligned} \tag{4.7}$$

Since ψ and $\nabla_{\mathbb{G}}\psi \in L^1(\mathbb{G})$ (see Remark 4.5), we have

$$\begin{aligned}\omega_p(y) &\lesssim \int_0^\infty \chi_{|y|\geq t} t\|(-\Delta_b)^{\frac{1}{2}}u\|_{L^p}\frac{\mathrm{d}t}{t} + \int_0^\infty \chi_{|y|\leq t}|y|\|(-\Delta_b)^{\frac{1}{2}}u\|_{L^p}\frac{\mathrm{d}t}{t}\\ &\lesssim \int_0^\infty \chi_{|y|\geq t} t\|\nabla_{\mathbb{G}}u\|_{L^p}\frac{\mathrm{d}t}{t} + \int_0^\infty \chi_{|y|\leq t}|y|\|\nabla_{\mathbb{G}}u\|_{L^p}\frac{\mathrm{d}t}{t}.\end{aligned}$$

Here, we used in the second inequality the continuity of the Riesz transform from L^p to L^p which gives $\|(-\Delta_b)^{\frac{1}{2}}u\|_{L^p} \lesssim \|\nabla_{\mathbb{G}}u\|_{L^p}$ (see [9]). Hence,

$$\begin{aligned}\left(\int_{\mathbb{G}}\left(\frac{\omega_p(y)}{|y|^s}\right)^q\frac{\mathrm{d}y}{|y|^Q}\right)^{\frac{1}{q}} &\lesssim \left(\int_{\mathbb{G}}\left(\int_0^\infty \frac{t}{|y|^s}\chi_{t\leq|y|}\|\nabla_{\mathbb{G}}u\|_{L^p}\frac{\mathrm{d}t}{t}\right)^q\frac{\mathrm{d}y}{|y|^Q}\right)^{\frac{1}{q}}\\ &\quad+\left(\int_{\mathbb{G}}\left(\int_0^\infty \frac{|y|}{|y|^s}\chi_{t\geq|y|}\|\nabla_{\mathbb{G}}u\|_{L^p}\frac{\mathrm{d}t}{t}\right)^q\frac{\mathrm{d}y}{|y|^Q}\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_0^\infty t^q\|\nabla_{\mathbb{G}}u\|_{L^p}^q\left(\int_{|y|\geq t}\frac{1}{|y|^{Q+s}}\,\mathrm{d}y\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\\ &\quad+\left(\int_0^\infty \|\nabla_{\mathbb{G}}u\|_{L^p}^q\left(\int_{|y|\leq t}\frac{1}{|y|^{Q-(1-s)}}\,\mathrm{d}y\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_0^\infty (t^{1-s}\|\nabla_{\mathbb{G}}u\|_{L^p})^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\end{aligned}\tag{4.8}$$

using the same trick of Lemma 3.3. Let us now move to the second equivalence. We will prove the direct inequality and the reverse one works exactly as in the previous setting. The main difference in this second equivalence is the fact that $\int_{\mathbb{G}}\partial_t p_\alpha(t,y)\,\mathrm{d}y \neq 0$ so we need to do a few more manipulations in order to have a similar setting as before. First recall from [22] that p_α satisfies the equation

$$\partial_t(t^{1-2\alpha}\partial_t p_\alpha) + t^{1-2\alpha}\Delta_b p_\alpha = 0 \quad \text{for } t > 0.$$

Hence, we have

$$\int_{\mathbb{G}}\partial_t(t^{1-2\alpha}\partial_t p_\alpha)(t,y)\,\mathrm{d}y = 0.$$

Thus,

$$\begin{aligned}t^{1-2\alpha}\partial_t u(t,x) &= \int_{\mathbb{G}} t^{1-2\alpha}\partial_t p_\alpha(t,y) f(xy)\,\mathrm{d}y\\ &= -\int_{\mathbb{G}}\int_t^\infty \partial_r(r^{1-2\alpha}\partial_r p_\alpha)(r,y)(f(xy)-f(x))\,\mathrm{d}r\,\mathrm{d}y\\ &= \int_{\mathbb{G}}\int_t^\infty r^{1-2\alpha}\Delta_b p_\alpha(r,y)(f(xy)-f(x))\,\mathrm{d}r\,\mathrm{d}y.\end{aligned}\tag{4.9}$$

It follows then that

$$
\begin{aligned}
t^{1-2\alpha}\|\partial_t u\|_{L^p} &\lesssim \int_{\mathbb{G}}\int_t^\infty r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\omega_p(y)\,\mathrm{d}r\,\mathrm{d}y\\
&\lesssim \int_{|y|<t}\int_t^\infty r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\omega_p(y)\,\mathrm{d}r\,\mathrm{d}y\\
&\quad+\int_{|y|>t}\int_t^\infty r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\omega_p(y)\,\mathrm{d}r\,\mathrm{d}y\\
&= I+II.
\end{aligned} \tag{4.10}
$$

We first estimate I. Using the fact that $|y| < t$, we have from Lemma 4.2, that

$$
r^{1-2\alpha}|\Delta_b p_\alpha(r,y)| \lesssim \frac{1}{r^{Q+1+2\alpha}}.
$$

Thus,

$$
I \lesssim \int_{|y|<t}\frac{1}{t^{Q+2\alpha}}\omega_p(y)\,\mathrm{d}y.
$$

In particular,

$$
\int_0^\infty [t^{2\alpha-s}I]^q\frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \left[\int_{\mathbb{G}}\chi_{|y|<t}\frac{1}{t^{Q+s}}\omega_p(y)\,\mathrm{d}y\right]^q\frac{\mathrm{d}t}{t}.
$$

We use then Lemma 3.3 with the same measure spaces as before, for the kernel $K(t,y) = \chi_{|y|<t}\frac{|y|^{Q+s}}{t^{Q+s}}$ and $f(y) = \frac{\omega_p(y)}{|y|^s}$. This leads to

$$
\left(\int_0^\infty [t^{2\alpha-s}I]^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}}.
$$

We move now to the second term. Indeed, we have

$$
\begin{aligned}
II &\lesssim \int_{|y|>t}\int_0^{|y|} r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\omega_p(y)\,\mathrm{d}r\,\mathrm{d}y\\
&\quad+\int_{|y|>t}\int_{|y|}^\infty r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\omega_p(y)\,\mathrm{d}r\,\mathrm{d}y.
\end{aligned} \tag{4.11}
$$

Now using again Lemma 4.2, we have that, for $r < |y|$,

$$
r^{1-2\alpha}|\Delta_b p_\alpha(r,y)| \lesssim \frac{r^{1-2\alpha}}{|y|^{Q+2}}.
$$

Therefore, since $\alpha < 1$, we have

$$
\int_0^{|y|} r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\,\mathrm{d}r \lesssim \frac{1}{|y|^{Q+2\alpha}}.
$$

Similarly, when $r > |y|$, using Lemma 4.2, we have

$$\begin{aligned}\int_{|y|}^{\infty} r^{1-2\alpha}|\Delta_b p_\alpha(r,y)|\,\mathrm{d}r &\lesssim \int_{|y|}^{\infty} r^{1-2\alpha}\frac{r^{2\alpha}}{(r^2+|y|^2)^{\frac{Q+2\alpha+2}{2}}}\,\mathrm{d}r\\ &\lesssim \int_{|y|}^{\infty}\frac{1}{r^{Q+2\alpha+1}}\,\mathrm{d}r\\ &\lesssim \frac{1}{|y|^{Q+2\alpha}}.\end{aligned} \tag{4.12}$$

Thus, we have

$$t^{2\alpha-s}II \lesssim \int_{\mathbb{G}} \chi_{|y|>t}\frac{t^{2\alpha-s}}{|y|^{Q+2\alpha}}\omega_p(y)\,\mathrm{d}y.$$

We use now Lemma 3.3 with the same measure spaces as before and $p = q$, for $K(t,y) = \chi_{|y|>t}\frac{t^{2\alpha-s}}{|y|^{2\alpha-s}}$ and $f(y) = \frac{\omega_p(y)}{|y|^s}$. Keeping in mind that the assumptions of Lemma 3.3 hold when $2\alpha > s$, we have

$$\left(\int_0^{\infty}[t^{2\alpha-s}II]^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}}.$$

Therefore, we conclude that

$$\left(\int_0^{\infty}[t^{1-s}\|\partial_t u\|_{L^p}]^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}}.$$

The proof of the last equivalence, is exactly similar to the first one, hence we omit it. □

5 Square function and BMO bounds

5.1 Square function bounds. The following square function bounds using the Sobolev norm will be useful later in the applications.

Theorem 5.1. *Let $f \in \mathcal{S}(\mathbb{G})$ and $1 < p < \infty$. Then:*
For $-Q < s < 1$ we have

$$\left\|\left(\int_0^{\infty}[t^{1-s}|\nabla_{\mathbb{G}}u(t,x)|]^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\right\|_{L^p} \lesssim \|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}.$$

For $s < 2\alpha$ we have

$$\left\|\left(\int_0^{\infty}[t^{1-s}|\frac{\partial}{\partial t}u(t,x)|]^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\right\|_{L^p} \lesssim \|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}.$$

For $-Q < s < 2$ we have

$$\left\|\left(\int_0^{\infty}[t^{2-s}|\nabla^2_{\mathbb{G}}u(t,x)|]^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\right\|_{L^p} \lesssim \|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}.$$

In order to proceed with the proof of Theorem 5.1, we first need to recall a few important properties of square functions. For further details, we refer the reader to [26, 45].

Let $\phi \in \mathcal{S}(\mathbb{G})$ be such that $\int_{\mathbb{G}} \phi \, \mathrm{d}x = 0$ and $\phi_t(x) = t^{-Q}\phi(\delta_{\frac{1}{t}}x)$. Then we define the square functions

$$S^{\beta}_{\phi}f(x) := \left(\int_0^{\infty} \int_{|x^{-1}y|<\beta t} |f * \phi_t(y)|^2 t^{-Q-1} \, \mathrm{d}y \, \mathrm{d}t \right)^{\frac{1}{2}} \tag{5.1}$$

where $\beta > 0$ and

$$g_{\phi}f(x) := \left(\int_0^{\infty} |f * \phi_t|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{5.2}$$

In [26], the authors proved the L^p boundedness of these operators. More precisely, they show that for $0 < p < \infty$, g_{ϕ} and S^{β}_{ϕ} are bounded from the Hardy space $H^p(\mathbb{G})$ to $L^p(\mathbb{G})$. From now on, we will write S_{ϕ} for S^{β}_{ϕ}. In order to use this result, we need to relax the assumption $\phi \in \mathcal{S}(\mathbb{G})$. We will need these bounds for some specific functions ϕ which are not in $\mathcal{S}(\mathbb{G})$.

Proposition 5.2. *Let $s > -1$ and $\phi(x) = \nabla_{\mathbb{G}}(-\Delta_b)^{\frac{s}{2}} p_{\alpha}(1, x)$ which implies the formula $\phi_t(x) = t^{1+s}\nabla_{\mathbb{G}}(-\Delta_b)^{\frac{s}{2}} p_{\alpha}(t, x)$. Then the function K^b_a defined by*

$$K^b_a(x) = \int_a^b \phi_t * \phi_t(x) \frac{\mathrm{d}t}{t}$$

converges as $a \to 0$ and $b \to \infty$ to a function K in $\mathcal{S}'(\mathbb{G})$ that is smooth on $\mathbb{G} \setminus \{0\}$ and homogeneous of degree $-Q$ around zero.

Proof. The formula $\phi_t(x) = t^{1+s}\nabla_{\mathbb{G}}(-\Delta_b)^{\frac{s}{2}} p_{\alpha}(t, x)$ follows from the homogeneity of p_{α}. Next notice that $p_{\alpha} * p_{\alpha}(t, x) = t^{-Q} p_{\alpha} * p_{\alpha}(1, \frac{x}{t})$. Indeed, this follows from the property that $f_t * g_t = (f * g)_t$. Here the convolution is only on the x variable, while the scaling is in the t variable. Now notice that

$$K^b_a(x) = \int_a^b t^{1+2s}(-\Delta_b)^{s+1}(p_{\alpha} * p_{\alpha})(t, x) \, \mathrm{d}t.$$

But then one can see, using Proposition 4.1 and an interpolation inequality of the form

$$\|(-\Delta_b)^{1+s}u\|_{L^{\infty}(\Omega)} \lesssim \|\Delta_b u\|^{\theta}_{L^{\infty}(\Omega)} \|(-\Delta_b)^2 u\|^{1-\theta}_{L^{\infty}(\Omega)}$$

with $\Omega = \{R \leq |x| \leq 2R\}$, that

$$|(-\Delta_b)^{s+1} p_{\alpha} * p_{\alpha}| \lesssim \begin{cases} |x|^{-(Q+2s+2)} & \text{if } t \leq |x|, \\ t^{-(Q+2s+2)} & \text{if } |x| \leq t. \end{cases}$$

Therefore, for $|x| > 0$,

$$t^{1+2s}(-\Delta_b)^{s+1}p_\alpha * p_\alpha(t,x) = O(t^{1+2s})$$

near zero and $t^{1+2s}(-\Delta_b)^{s+1}p_\alpha * p_\alpha(t,x) = O(t^{-Q-1})$ near ∞. Thus, as long as $1+2s > -1$, the integral converges absolutely to a smooth function on $\mathbb{G}\setminus\{0\}$. Moreover, if we let $K = \lim_{a\to 0; b\to\infty} K_a^b$, we have that $K(rx) = r^{-Q}K(x)$ which finishes the proof. □

One also has the same result for

$$\phi_t = \begin{cases} t^{1+s}\nabla_{\mathbb{G}}(-\Delta_b)^{\frac{s}{2}}p_\alpha & \text{if } s > -1,\\ t^{1+s}(-\Delta_b)^{\frac{s}{2}}\frac{\partial}{\partial t}p_\alpha & \text{if } s > -2\alpha,\\ t^{2+s}(-\Delta)^{\frac{s}{2}}(-\Delta_b)p_\alpha & \text{if } s > -2,\\ t^{2+s}\nabla_{\mathbb{G}}(-\Delta_b)^{\frac{s}{2}}\frac{\partial}{\partial t}p_\alpha & \text{if } s > -1-2\alpha. \end{cases} \tag{5.3}$$

Recall the square functions g_ϕ and $S_\phi^1 = S_\phi$ defined in (5.1) and (5.2).

Proposition 5.3. *Let ϕ_t be defined as in* (5.3)*. Then S_ϕ and g_ϕ are bounded from L^p to L^p, for* $1 < p < \infty$.

Proof. We will follow here the proof in [26] for the case $\phi \in \mathcal{S}(\mathbb{G})$ and we will present it for ϕ_t as in Proposition 5.2 since the proof is similar for the remaining functions in (5.3). Indeed, one first proves the L^2 bound, that is

$$\begin{aligned} \|g_\phi f\|_{L^2}^2 &= \int_{\mathbb{G}}\int_0^\infty f*\phi_t f*\phi_t \frac{\mathrm{d}t}{t} \\ &= \int_{\mathbb{G}} f*K(x)f(x)\,\mathrm{d}x \le \|K*f\|_{L^2}\|f\|_{L^2}. \end{aligned} \tag{5.4}$$

But, from Proposition 3.2, K is a kernel of type $(0,2)$, thus we have that

$$\|K*f\|_{L^2} \lesssim \|f\|_{L^2}.$$

Therefore

$$\|g_\phi\|_{L^2} \lesssim \|f\|_{L^2}.$$

We define the space $X = L^2((0,\infty), \frac{\mathrm{d}t}{t})$ and the X-valued distribution Φ defined for $f \in \mathcal{S}(\mathbb{G})$ by

$$\langle\Phi, f\rangle(t) = \int_G f(x)\phi_t(x)\,\mathrm{d}x.$$

We claim that this distribution is well defined. Indeed, we have

$$|\langle\Phi, f\rangle(t)| \le \frac{1}{t^Q}\|\phi\|_{L^\infty}\|f\|_{L^1}.$$

Next, we notice that since $\int \phi = 0$ we have that

$$|\langle \Phi, f\rangle(t)| \leq \int_{\mathbb{G}} |f(tx) - f(0)|\phi(x)\,\mathrm{d}x.$$

Since $f \in \mathcal{S}(\mathbb{G})$ we see that

$$t \mapsto \left|\frac{\langle \Phi, f\rangle(t)}{t}\right|$$

is bounded near zero, therefore $\langle \Phi, f\rangle \in L^2((0,\infty), \frac{\mathrm{d}t}{t})$. Hence, $g_\phi f(x)$ is well defined and

$$g_\phi f(x) = \|f * \Phi\|_X.$$

And so far we have proved that g_ϕ is bounded from L^2 to L^2_X. Moreover, if we look at $\Phi(x)(t) = \phi_t(x)$ we have that

$$\begin{aligned}\|D^\beta \Phi(x)\|_X^2 &= \int_0^\infty |t^{1+s} D^{\beta+1}(-\Delta_b)^{\frac{s}{2}} p_\alpha(t,x)|^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^{|x|} t^{1+2s}|x|^{-2(Q+\beta+s+1)}\,\mathrm{d}t + \int_{|x|}^\infty t^{1+2s} t^{-2(Q+\beta+1+s)}\,\mathrm{d}t \\ &\lesssim |x|^{-2(Q+\beta)}.\end{aligned}$$

Hence Φ is an X-valued kernel of type $(0,r)$ for al $r > 0$ which leads to the boundedness of $f * \Phi$ from L^p to L^p_X for $1 < p < \infty$. Thus

$$\|g_\phi f\|_{L^p} \lesssim \|f\|_{L^p}.$$

A similar bound holds for the operator S^β_ϕ. □

Proof of Theorem 5.1. The proof of Theorem 5.1 now is a straightforward consequence of Proposition 5.3. First, we write

$$t^{1-s}\nabla_{\mathbb{G}} u(t,x) = t^{1-s}\nabla_{\mathbb{G}} p_\alpha * f = t^{1-s}\nabla_{\mathbb{G}}(-\Delta_b)^{-\frac{s}{2}} p_\alpha * (-\Delta_b)^{\frac{s}{2}} f.$$

Applying Proposition 5.3, we have the desired result for $s < 1$. □

5.2 BMO bounds. Next we provide some equivalent characterizations of the BMO norm that will be useful in the coming applications. First, given a function $f \in L^1_{loc}(\mathbb{G})$ and B a ball in $\mathbb{G}$, we define

$$m_B = \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x.$$

Let $\mathcal{B}$ be the collection of all the balls in $\mathbb{G}$. A function f is said to be in BMO if

$$[f]_{\mathrm{BMO}} = \sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B |f(x) - m_B|\,\mathrm{d}x < \infty.$$

We recall next the characterization of the BMO norm using the Carleson measure. If we denote

$$T(B_r(x_0)) = \{(t,x) \in \mathbb{R}^+ \times \mathbb{G} : |x_0^{-1}x| < r - t\},$$

then we have the following proposition [45].

Proposition 5.4. *Let $f \in \mathcal{S}(\mathbb{G})$ and $\phi \in \mathcal{S}(\mathbb{G})$ be such that $\int_{\mathbb{G}} \phi(x)\,\mathrm{d}x = 0$. Then*

$$\sup_{B\in\mathcal{B}} \frac{1}{|B|} \int_{T(B)} |f * \phi_t|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{t} \lesssim [f]_{\mathrm{BMO}}^2.$$

If we assume the existence of $\psi \in \mathcal{S}(\mathbb{G})$, with $\int_{\mathbb{G}} \psi\,\mathrm{d}x = 0$ and $\int_0^\infty \phi_t * \psi_t \frac{\mathrm{d}t}{t} = \tilde{\delta}_0$, then

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |f * \phi_t|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{t} \right)^{\frac{1}{2}} \approx [f]_{\mathrm{BMO}}.$$

From the equivalence stated above, one gets the following proposition.

Proposition 5.5. *Let $f \in \mathcal{S}(\mathbb{G})$. Then for ϕ_t defined in* (5.3)*, we have*

$$[f]_{\mathrm{BMO}} \approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |f * \phi_t|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

From the previous proposition we get in particular

$$\begin{aligned}
[f]_{\mathrm{BMO}} &\approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |t\nabla_{\mathbb{G}} u(t,x)|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |t^2 \Delta_{\mathbb{G}} u(t,x)|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} \left| t\frac{\partial}{\partial t} u(t,x) \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},
\end{aligned}$$

and in the fractional setting

$$\begin{aligned}
{}_{\mathrm{BMO}} &\approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |t^s (-\Delta_b)^{\frac{s}{2}} u(t,x)|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\approx \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_{T(B)} |t^{1+s} \nabla_{\mathbb{G}} (-\Delta_b)^{\frac{s}{2}} u(t,x)|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.5}$$

We finish now by recalling the following duality result between the Carleson measure and the square function [45].

Lemma 5.6. *Let $G, F : \mathbb{R}^+ \times \mathbb{G} \to \mathbb{R}$ be two functions. Then*

$$\int_{\mathbb{G}} \int_0^\infty F(t,x)G(t,x)\frac{\mathrm{d}t}{t}\,\mathrm{d}x$$
$$\lesssim \sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_{T(B)} |F(t,y)|^2 \frac{\mathrm{d}t}{t}\,\mathrm{d}y\right)^{\frac{1}{2}} \int_{\mathbb{G}} \left(\int_{|y^{-1}x|<t} |G(t,y)|^2 \frac{\mathrm{d}t}{t^{Q+1}}\,\mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}x,$$

whenever the right-hand side is finite.

Corollary 5.7. *Let $G : \mathbb{R}^+ \times \mathbb{G} \to \mathbb{R}$ such that*

$$\int_{\mathbb{G}} \left(\int_{|y^{-1}x|<t} |G(t,y)|^2 \frac{\mathrm{d}t}{t^{Q+1}}\,\mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}x < \infty.$$

Then

$$\int_{\mathbb{G}} \int_0^\infty \left(t\frac{\partial}{\partial t}u(t,x)\right)G(t,x)\frac{\mathrm{d}t}{t}\,\mathrm{d}x \lesssim [f]_{\mathrm{BMO}} \int_{\mathbb{G}} \left(\int_{|y^{-1}x|<t} |G(t,y)|^2 \frac{\mathrm{d}t}{t^{Q+1}}\,\mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}x.$$

This inequality still holds if we replace $t\frac{\partial}{\partial t}u(t,x)$ by any one of:

- $t\nabla_{\mathbb{G}}u(t,x)$,
- $t^2\Delta_b u(t,x)$,
- $t^s(-\Delta_b)^{\frac{s}{2}}u(t,x)$,
- $t^{1+s}\tilde{\nabla}(-\Delta_b)^{\frac{s}{2}}u(t,x)$.

6 Applications

Before starting this section we recall some relevant maximal function bounds. Given a function $\phi \colon \mathbb{G} \to \mathbb{R}$ satisfying the growth condition

$$|\phi(x)| \lesssim \frac{1}{(1+|x|)^\lambda},$$

for a given $\lambda > 0$, one can define the two maximal functions

$$(\mathcal{M}^0_\phi f)(x) = \sup_{t>0}(f * \phi_t)(x)$$

and

$$(\mathcal{M}_\phi f)(x) = \sup\{|f * \phi_t| : |x^{-1}y| < t, 0 < t < \infty\}.$$

With these definitions, one has the following theorem [26].

Theorem 6.1. *For $\lambda > Q$, $\mathcal{M}^0_\phi$ and $\mathcal{M}_\phi$ are bounded from $L^p(\mathbb{G})$ to $L^p(\mathbb{G})$ for $p > 1$ and from $L^1(\mathbb{G})$ to weak $L^1(\mathbb{G})$.*

In this section, if $f \in \mathcal{S}(\mathbb{G})$ we will write $F_\alpha = f * p_\alpha$. We also use the notation $\tilde{\nabla} = \nabla_{\mathbb{G}} \oplus \frac{\partial}{\partial t}$, defined by $\tilde{\nabla}u = (\nabla_{\mathbb{G}}u, \frac{\partial u}{\partial t})$. In what follows, we will write $\mathcal{M}$ and $\mathcal{M}^0$ instead of $\mathcal{M}_\phi$ and $\mathcal{M}^0_\phi$, since the function ϕ will be different depending on the situation.

6.1 Integral inequalities.

Theorem 6.2. *Let* $f, g, h \in \mathcal{S}(\mathbb{G})$ *and* $1 < p_1, p_2, p_3 < \infty$ *such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.$$

Then:

(1) *For* $s_1, s_2 \in (0, \min\{1, 2\alpha\})$ *and* $Q > s_3 \geq 0$,

$$\int_{\mathbb{R}^+ \times \mathbb{G}} t^{2-s_1-s_2+s_3} |\tilde{\nabla} F_\alpha| |\tilde{\nabla} G_\alpha| |H_\alpha| \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \|(-\Delta_b)^{\frac{s_1}{2}} f\|_{L^{p_1}} \|(-\Delta_b)^{\frac{s_2}{2}} g\|_{L^{p_2}} \|I_{s_3} h\|_{L^{p_3}}.$$

(2) *For* $s_1 \in (0, \min\{1, 2\alpha\})$ *and* $Q > s_3, s_2 \geq 0$,

$$\int_{\mathbb{R}^+ \times \mathbb{G}} t^{2-s_1+s_2+s_3} |\tilde{\nabla} F_\alpha| |\tilde{\nabla} G_\alpha| |H_\alpha| \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \|(-\Delta_b)^{\frac{s_1}{2}} f\|_{L^{p_1}} \|I_{s_2} g\|_{L^{p_2}} \|I_{s_3} h\|_{L^{p_3}}.$$

(3) *For* $s_1 \in (0, \min\{1 + 2\alpha, 2\})$, $s_2 \in (0, \min\{1, 2\alpha\})$ *and* $0 \leq s_3 < Q$,

$$\int_{\mathbb{R}^+ \times \mathbb{G}} t^{3-s_1+s_2+s_3} |\nabla_{\mathbb{G}} \tilde{\nabla} F_\alpha| |\tilde{\nabla} G_\alpha| |H_\alpha| \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \|(-\Delta_b)^{\frac{s_1}{2}} f\|_{L^{p_1}} \|(-\Delta_b)^{\frac{s_2}{2}} g\|_{L^{p_2}} \|I_{s_3} h\|_{L^{p_3}}.$$

Here I_α *is the fractional integration of order* α, *that is* $I_\alpha u = (-\Delta_b)^{\frac{\alpha}{2}} u$.

Proof. We will present the proof of (1). The proofs of (2) and (3) follow the same idea. We will apply the result of Theorem 6.1 for $\phi_t = t^s(-\Delta_b)^{\frac{s}{2}} p_\alpha$. Indeed, we have that

$$(-\Delta_b)^{\frac{s}{2}} p_\alpha(1, x) \lesssim \frac{1}{(1 + |x|)^{Q+s}}.$$

Hence, we have by definition of $\mathcal{M}^0_\phi$ that

$$\sup_{t>0} t^s |H_\alpha| = \sup_{t>0} |t^s (-\Delta_b)^{\frac{s}{2}} p_\alpha * I_s f| = \mathcal{M}^0_\phi(I_s f).$$

Hence,

$$\int_{\mathbb{G}} \int_0^\infty t^{2-s_1-s_2+s_3} |\tilde{\nabla} F_\alpha| |\tilde{\nabla} G_\alpha| |H_\alpha| \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \int_{\mathbb{G}} \mathcal{M}^0_\phi(I_{s_3} h) \int_0^\infty t^{1-s_1} |\tilde{\nabla} F_\alpha| t^{1-s_2} |\tilde{\nabla} G_\alpha| \frac{\mathrm{d}t}{t}\,\mathrm{d}x.$$

The result then follows from Hölder's inequality and Theorem 5.1. □

Theorem 6.3. *Let $f, g, h \in \mathcal{S}(\mathbb{G})$, and $\frac{1}{p}+\frac{1}{q}=1$ with $1<p<\infty$. Then for $s\in(0,1)$,*

$$\int_{\mathbb{R}^+\times\mathbb{G}} t^{2+2(1-s)}|\tilde{\nabla}H_\alpha||\nabla_{\mathbb{G}}\tilde{\nabla}F_\alpha||\tilde{\nabla}G_\alpha|\frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim [h]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}\|(-\Delta_b)^{\frac{s}{2}}g\|_{L^q},$$

and for $s<2\alpha$ we have

$$\int_{\mathbb{R}^+\times\mathbb{G}} t^{2-s}|\tilde{\nabla}H_\alpha|\left|\frac{\partial}{\partial t}F_\alpha\right||G_\alpha|\frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim [h]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}\|g\|_{L^q}.$$

Proof. Let us again start by proving the first claim. Indeed, using Corollary 5.7 we have that

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{G}} t^{2+2(1-s)}|\tilde{\nabla}H_\alpha||\nabla_{\mathbb{G}}\tilde{\nabla}F_\alpha||\tilde{\nabla}G_\alpha|\frac{\mathrm{d}x\,\mathrm{d}t}{t}\\
&\lesssim [h]_{\mathrm{BMO}}\int_{\mathbb{G}}\left(\int_{|y^{-1}x|<t}(t^{1+2(1-s)}|\nabla_{\mathbb{G}}\tilde{\nabla}F_\alpha||\tilde{\nabla}G_\alpha|)^2\frac{\mathrm{d}t}{t^{Q+1}}\right)^{\frac{1}{2}}\mathrm{d}x \qquad (6.1)\\
&\lesssim [h]_{\mathrm{BMO}}\int_{\mathbb{G}}\mathcal{M}((-\Delta_b)^{\frac{s}{2}}f)(x)S^1_\phi((-\Delta_b)^{\frac{s}{2}}g)(x)\,\mathrm{d}x\\
&\lesssim [h]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}f\|_{L^p}\|(-\Delta_b)^{\frac{s}{2}}g\|_{L^q}.
\end{aligned}$$

A similar proof holds for the second claim. □

6.2 Three-term commutator. Let $u, v\in\mathcal{S}(\mathbb{G})$. Then we define the **three-term commutator** $\mathcal{H}_\alpha(u,v)$ by

$$\mathcal{H}_\alpha(u,v) := (-\Delta_b)^\alpha(uv) - u(-\Delta_b)^\alpha v - v(-\Delta_b)^\alpha u.$$

This commutator was studied in the Euclidean setting in [35, 44] and in the case of Carnot groups in [36]. We want also to point out that one can obtain easy bounds for this commutator in Besov spaces. Indeed

Proposition 6.4. *Let $\alpha\in(0,1)$, assume that $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{q_1}+\frac{1}{q_2}=1$. We let $s_1, s_2>0$ so that $s_1+s_2=2\alpha$. Then we have*

$$\|\mathcal{H}_\alpha(u,v)\|_{L^r}\lesssim \|u\|_{\dot{B}^{s_1}_{p_1,q_1}}\|v\|_{\dot{B}^{s_2}_{p_2,q_2}}.$$

Proof. The proof follows directly from the pointwise expression of the commutator. Using

$$(-\Delta_b)^\alpha u(x) = \int_{\mathbb{G}}(u(x)-u(y))\tilde{R}_\alpha(xy^{-1})\,\mathrm{d}y,$$

we can write, as in [36],

$$\mathcal{H}_\alpha(u,v)(x) = \int_{\mathbb{G}} [u(xy)-u(x)][v(xy)-v(x)]\tilde{R}_\alpha(y)\,\mathrm{d}y.$$

Since $\tilde{R}_\alpha \approx |y|^{-Q-2\alpha}$, we have for $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}$, $1=\frac{1}{q_1}+\frac{1}{q_2}$ and $2\alpha = s_1+s_2$:

$$\begin{aligned}
\|\mathcal{H}_\alpha(u,v)\|_{L^r} &\lesssim \left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} \frac{[u(xy)-u(x)][v(xy)-v(x)]}{|y|^{2\alpha}}\frac{\mathrm{d}y}{|y|^Q}\right)^r \mathrm{d}x\right)^{\frac{1}{r}} \\
&\lesssim \int_{\mathbb{G}}\left(\int_{\mathbb{G}}\Big(\frac{[u(xy)-u(x)][v(xy)-v(x)]}{|y|^{2\alpha}}\Big)^r \mathrm{d}x\right)^{\frac{1}{r}}\frac{\mathrm{d}y}{|y|^Q} \\
&\lesssim \int_{\mathbb{G}} \frac{1}{|y|^{2\alpha}}\|u(xy)-u(x)\|_{L^{p_1}}\|v(xy)-v(x)\|_{L^{p_2}}\frac{\mathrm{d}y}{|y|^Q} \\
&\lesssim \|u\|_{\dot{B}^{s_1}_{p_1,q_1}}\|v\|_{\dot{B}^{s_2}_{p_2,q_2}}.
\end{aligned}$$

□

The case of L^p spaces is a little bit more difficult and technical as in [36]. We will see here that for some range of α, we can obtain a relatively simple proof of some of these bounds and in fact extend the range of the estimates proved in [36] to include a BMO-type estimate.

6.2.1 L^p-type estimates.

Theorem 6.5. *Let $\alpha \in (0,\frac{1}{2}]$. Then one has*

$$\|\mathcal{H}_\alpha(u,v)\|_{L^p} \lesssim \|(-\Delta_b)^\alpha u\|_{L^p}[v\ ]_{\mathrm{BMO}}.$$

Moreover, for $\alpha=\alpha_1+\alpha_2$ with $\alpha_1,\alpha_2 \in (0.\frac{1}{2})$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, one has

$$\|\mathcal{H}_\alpha(u,v)\|_{L^p} \lesssim \|(-\Delta_b)^{\alpha_1}u\|_{L^{p_1}}\|(-\Delta_b)^{\alpha_1}v\|_{L^{p_2}}.$$

Proof. We will start first by proving the second claim. We let $h \in L^{p'}$, and we propose to estimate $\int_{\mathbb{G}}\mathcal{H}_\alpha(u,v)h\,\mathrm{d}x$. Using the fact that

$$\lim_{t\to 0}\left(t^{1-2\alpha}\frac{\partial}{\partial t}U_\alpha(t,x)\right) = c_\alpha(-\Delta_b)^\alpha u(x)$$

and that

$$\frac{\partial}{\partial t}\left(t^{1-2\alpha}\frac{\partial}{\partial t}U_\alpha\right) = -t^{1-2\alpha}\Delta_b U_\alpha$$

we have that

$$\begin{aligned}
\left|\int_{\mathbb{G}}\mathcal{H}_\alpha(u,v)h\,\mathrm{d}x\right| &\approx \left|\int_{\mathbb{G}}\int_0^\infty \partial_t[t^{1-2\alpha}(U_\alpha V_\alpha\partial_t H_\alpha - U_\alpha H_\alpha\partial_t V_\alpha - V_\alpha H_\alpha \partial_t U)]\,\mathrm{d}t\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-2\alpha}[2\partial_t U_\alpha\partial_t V_\alpha + \nabla_{\mathbb{G}}V_\alpha\nabla_{\mathbb{G}}U_\alpha]H_\alpha\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\lesssim \int_{\mathbb{G}\times\mathbb{R}^+} t^{2-2\alpha}|\tilde{\nabla}U_\alpha||\tilde{\nabla}V_\alpha||H_\alpha|\frac{\mathrm{d}x\,\mathrm{d}t}{t}.
\end{aligned}$$

Now using Theorem 6.2, we have that

$$\left|\int_{\mathbb{G}} \mathcal{H}_\alpha(u,v)h\,\mathrm{d}x\right| \lesssim \|(-\Delta_b)^{\alpha_1}u\|_{L^{p_1}}\|(-\Delta_b)^{\alpha_2}v\|_{L^{p_2}}\|h\|_{L^{p'}}.$$

Notice that this also provides the proof of the first claim for $\alpha < \frac{1}{2}$ using Theorem 6.3. It remains thus to treat the case $\alpha = \frac{1}{2}$. In this case, we have after another integration by parts

$$\begin{aligned}\left|\int_{\mathbb{G}} \mathcal{H}_\alpha(u,v)h\,\mathrm{d}x\right| &\approx \left|\int_{\mathbb{G}\times\mathbb{R}^+} [2\partial_t U_\alpha \partial_t V_\alpha + \nabla_{\mathbb{G}} V_\alpha \nabla_{\mathbb{G}} U_\alpha] H_\alpha\,\mathrm{d}x\,\mathrm{d}t\right| \\ &= \left|\int_{\mathbb{G}\times\mathbb{R}^+} t\frac{\partial}{\partial t}[(2\partial_t U_\alpha \partial_t V_\alpha + \nabla_{\mathbb{G}} V_\alpha \nabla_{\mathbb{G}} U_\alpha) H_\alpha]\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\lesssim \int_{\mathbb{G}\times\mathbb{R}^+} t[|\tilde{\nabla} U_\alpha||\tilde{\nabla} V_\alpha||\tilde{\nabla} H_\alpha| + |\tilde{\nabla} V_\alpha||\tilde{\nabla}\nabla_{\mathbb{G}} U_\alpha||H_\alpha|]\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Next, writing $t^2 = tt^0t$ for the first term and $t^2 = tt^{2-2(\frac{1}{2})}$ we get

$$\left|\int_{\mathbb{G}} \mathcal{H}_\alpha(u,v)h\,\mathrm{d}x\right| \lesssim [v\ ]_{\mathrm{BMO}}\|h\|_{L^{p'}}\|(-\Delta_b)^{\frac{1}{2}}u\|_{L^p}.$$

□

6.2.2 Rivière–Da Lio three-term commutator.

Theorem 6.6. *If* $2\alpha \leq 1$ *we have*

$$\|(-\Delta_b)^\alpha \mathcal{H}_\alpha(u,v)\|_{H^1} \lesssim \|(-\Delta_b)^\alpha u\|_{L^p}\|(-\Delta_b)^\alpha u\|_{L^{p'}}.$$

Proof. Here we will use the duality between H^1 and BMO (see [26]).

Case $2\alpha < 1$**.** Let $h \in \mathcal{S}(\mathbb{G})$ and $g = (-\Delta_b)^\alpha h$. Then we have

$$\begin{aligned}\left|\int_{\mathbb{G}} \mathcal{H}_\alpha(u,v)g\,\mathrm{d}x\right| &= \left|\int_{\mathbb{G}} uv(-\Delta_b)^\alpha g - ug(-\Delta_b)^\alpha v - vg(-\Delta_b)^\alpha u\,\mathrm{d}x\right| \\ &= \left|\int_{\mathbb{G}\times\mathbb{R}^+} \partial_t(t^{1-2\alpha}(U_\alpha V_\alpha \partial_t G_\alpha - U_\alpha G_\alpha \partial_t V_\alpha - V_\alpha G_\alpha \partial_t U_\alpha))\,\mathrm{d}t\,\mathrm{d}x\right| \\ &\lesssim \int_{\mathbb{G}\times\mathbb{R}^+} t^{1-2\alpha}|\nabla_{\mathbb{G}} U_\alpha||\nabla_{\mathbb{G}} V_\alpha||G_\alpha|\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \left|\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-2\alpha}[\partial_t U_\alpha \partial_t V_\alpha G_\alpha]\,\mathrm{d}t\,\mathrm{d}x\right|.\end{aligned}$$

We write the first term as $t^{1-2\alpha}|\nabla_{\mathbb{G}} U_\alpha| t^{1-2\alpha}|\nabla_{\mathbb{G}} V_\alpha| t^{2\alpha}|G_\alpha|\frac{1}{t}$ to get, using Theorem 6.3, an estimate of the form

$$\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-2\alpha}|\nabla_{\mathbb{G}} U_\alpha||\nabla_{\mathbb{G}} V_\alpha||G_\alpha|\,\mathrm{d}t\,\mathrm{d}x \lesssim [g]_{\mathrm{BMO}}\|(-\Delta_b)^\alpha u\|_{L^p}\|(-\Delta_b)^\alpha v\|_{L^q}.$$

The second term is a little more involved as in the proof of Theorem 6.5: an extra integration by parts is needed. Indeed,

$$\begin{aligned}\int_{\mathbb{G}\times\mathbb{R}} t^{1-2\alpha}[\partial_t U_\alpha \partial_t V_\alpha G_\alpha] &= \frac{1}{2\alpha}\int_{\mathbb{G}\times\mathbb{R}^+} t^{2\alpha}\partial_t(t^{1-2\alpha}\partial_t U_\alpha t^{1-2\alpha}\partial_t V_\alpha G_\alpha)\,\mathrm{d}t\,\mathrm{d}x\\ &= \frac{-1}{2\alpha}\int_{\mathbb{G}\times\mathbb{R}^+} t^{2-2\alpha}(\Delta_b U_\alpha \partial_t V_\alpha G_\alpha + \Delta_b V_\alpha U_\alpha G_\alpha\,\mathrm{d}t\,\mathrm{d}x\\ &\quad+\frac{1}{2\alpha}\int_{\mathbb{G}\times\mathbb{R}^+} t^{2-2\alpha}\partial_t U_\alpha \partial_t V_\alpha \partial_t G_\alpha\,\mathrm{d}t\,\mathrm{d}x\\ &= \frac{-1}{2\alpha}\int_{\mathbb{G}\times\mathbb{R}^+} t^{2-2\alpha}(\Delta_b U_\alpha \partial_t V_\alpha G_\alpha + \Delta_b V_\alpha U_\alpha G_\alpha\,\mathrm{d}t\,\mathrm{d}x\\ &\quad+\frac{1}{8\alpha}\int_{\mathbb{G}\times\mathbb{R}^+} t^{4\alpha}\partial_t(t^{1-2\alpha}\partial_t U_\alpha t^{1-2\alpha}\partial_t V_\alpha t^{1-2\alpha}\partial_t G_\alpha)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now the first two terms can be easily bounded by the desired quantity. It remains to bound the last one:

$$\begin{aligned}&\int_{\mathbb{G}\times\mathbb{R}^+} t^{4\alpha}\partial_t(t^{1-2\alpha}\partial_t U_\alpha t^{1-2\alpha}\partial_t V_\alpha t^{1-2\alpha}\partial_t G_\alpha)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{\mathbb{G}\times\mathbb{R}^+} t^{3-2\alpha}\Big[(\Delta_b U_\alpha \partial_t V_\alpha + \Delta_b V_\alpha \partial_t U_\alpha)\partial_t G_\alpha\\ &\qquad - (\partial_t \nabla_{\mathbb{G}} U_\alpha \partial_t V_\alpha + \partial_t \nabla_{\mathbb{G}} V_\alpha \partial_t U_\alpha)\nabla_{\mathbb{G}} G_\alpha\Big]\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Again all the terms here have the right form of Theorems 6.2 and 6.3 and they provide the desired bound.

Case $2\alpha = 1$**.** In this case, we have

$$\begin{aligned}&\left|\int_{\mathbb{G}} \mathcal{H}_{\frac{1}{2}}(u,v)g\,\mathrm{d}x\right|\\ &= \left|\int_{\mathbb{G}\times\mathbb{R}^+} \partial_t(U_{\frac12}V_{\frac12}(-\Delta_b)H_{\frac12} - \partial_t U_{\frac12}V_{\frac12}\partial_t H_{\frac12} - \partial_t V_{\frac12}U_{\frac12}\partial_t H_{\frac12})\,\mathrm{d}t\,\mathrm{d}x\right|\\ &= \left|\int_{\mathbb{G}\times\mathbb{R}^+} t\partial_t((U_{\frac12}V_{\frac12})(-\Delta_b)H_{\frac12} - (U_{\frac12}V_{\frac12})_{tt}\partial_t H_{\frac12})\,\mathrm{d}t\,\mathrm{d}x\right|\\ &= \left|\int_{\mathbb{G}\times\mathbb{R}^+} t\partial_t(\tilde{\Delta}_b(U_{\frac12}V_{\frac12})\partial_t H_{\frac12})\,\mathrm{d}t\,\mathrm{d}x\right|\\ &= 2\left|\int_{\mathbb{G}\times\mathbb{R}^+} t\partial_t(\tilde{\nabla}U_{\frac12}\tilde{\nabla}V_{\frac12}\partial_t H_{\frac12})\,\mathrm{d}t\,\mathrm{d}x\right|\\ &\lesssim \int_{\mathbb{G}\times\mathbb{R}^+} t|\partial_t(\tilde{\nabla}U_{\frac12}\tilde{\nabla}V_{\frac12})||\partial_t H_{\frac12})|\,\mathrm{d}t\,\mathrm{d}x\\ &\qquad+\int_{\mathbb{G}\times\mathbb{R}^+} t|\nabla_{\mathbb{G}}(\tilde{\nabla}U_{\frac12}\tilde{\nabla}V_{\frac12})||\nabla_{\mathbb{G}} H_{\frac12}|\,\mathrm{d}t\,\mathrm{d}x,\end{aligned}$$

where here $\tilde{\Delta}_b = \Delta_b + \partial_{tt}$ and we used the harmonicity of the extension. Notice here that we can finish the proof as in the previous case. □

Note that one can also capture the $(L^p, L^q) \to L^r$-type estimates in [36] by slightly modifying the proof and using the L^p estimates for the Riesz potential.

6.3 Chanillo-type commutator. We recall here the commutator estimate proved by Chanillo [18]:

$$\|[I_s, v\ ]u\|_{L^p} \lesssim \|u\|_{L^q}[v\ ]_{\mathrm{BMO}}.$$

Notice that

$$[I_s, v\ ]u = I_s(uv) - vI_s(u).$$

Therefore, if we set $u = (-\Delta_b)^{\frac{s}{2}}a$, we have

$$\int_{\mathbb{G}} (-\Delta_b)^{\frac{s}{2}}([I_s, v\ ]u)h = \int_{\mathbb{G}} v\ [(-\Delta_b)^{\frac{s}{2}}a - a(-\Delta_b)^{\frac{s}{2}}h]\,\mathrm{d}x.$$

So we propose to estimate an integral of the form

$$\int_{\mathbb{G}} v\ [(-\Delta_b)^{\frac{s}{2}}u - u(-\Delta_b)^{\frac{s}{2}}h]\,\mathrm{d}x.$$

Theorem 6.7. *For* $\frac{1}{p} + \frac{1}{r} - \frac{s}{Q} = 1$, *we have*

$$\left|\int_{\mathbb{G}} v\ [(-\Delta_b)^{\frac{s}{2}}u - u(-\Delta_b)^{\frac{s}{2}}h]\,\mathrm{d}x\right| \lesssim [v\ ]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}u\|_{L^p}\|\,\|(-\Delta_b)^{\frac{s}{2}}h\|_{L^r}.$$

Proof. Again, we use the same trick, that is we write

$$\begin{aligned}
&\left|\int_{\mathbb{G}} v\ [h(-\Delta_b)^{\frac{s}{2}}u - u(-\Delta_b)^{\frac{s}{2}}h]\,\mathrm{d}x\right| \\
&\qquad\approx \left|\int_{\mathbb{G}\times\mathbb{R}^+} \partial_t[t^{1-s}(\partial_t U_{\frac{s}{2}} H_{\frac{s}{2}} - \partial_t H_{\frac{s}{2}} U_{\frac{s}{2}})V_{\frac{s}{2}}]\,\mathrm{d}t\,\mathrm{d}x\right| \\
&\qquad\lesssim \left|\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-s}(\nabla_{\mathbb{G}} U_{\frac{s}{2}} H_{\frac{s}{2}} - \nabla_{\mathbb{G}} U_{\frac{s}{2}} H_{\frac{s}{2}})\nabla_{\mathbb{G}} V_{\frac{s}{2}}\,\mathrm{d}t\,\mathrm{d}x\right| \\
&\qquad\quad + \left|\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-s}(\partial_t U_{\frac{s}{2}} H_{\frac{s}{2}} - \partial_t U_{\frac{s}{2}} H_{\frac{s}{2}})\partial_t V_{\frac{s}{2}}\,\mathrm{d}t\,\mathrm{d}x\right|.
\end{aligned}$$

Using Theorem 6.3, we have that

$$\begin{aligned}
&\left|\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-s}(\nabla_{\mathbb{G}} U_{\frac{s}{2}} H_{\frac{s}{2}} - \nabla_{\mathbb{G}} U_{\frac{s}{2}} H_{\frac{s}{2}})\nabla_{\mathbb{G}} V_{\frac{s}{2}}\,\mathrm{d}t\,\mathrm{d}x\right| \\
&\qquad\lesssim [v\ ]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}u\|_{L^p}\|h\|_{L^q} \\
&\qquad\lesssim [v\ ]_{\mathrm{BMO}}\|(-\Delta_b)^{\frac{s}{2}}u\|_{L^p}\|(-\Delta_b)^{\frac{s}{2}}h\|_{L^r}
\end{aligned}$$

where the second inequality follows from the Sobolev embeddings with $\frac{1}{r} = \frac{1}{q} + \frac{s}{Q}$. The second term, on the other hand, cannot be bounded directly since we are in the case $s = 2\alpha$. That is why we perform another integration by parts:

$$\begin{aligned}
&\int_{\mathbb{G}\times\mathbb{R}^+} t^{1-s}(\partial_t U_{\frac{s}{2}} H_{\frac{s}{2}} - \partial_t U_{\frac{s}{2}} H_{\frac{s}{2}})\partial_t V_{\frac{s}{2}} \,\mathrm{d}t\,\mathrm{d}x \\
&\quad = \frac{1}{s}\int_{\mathbb{G}\times\mathbb{R}^+} t^s \partial_t (t^{1-s}(\partial_f U_{\frac{s}{2}} H_{\frac{s}{2}} - U_{\frac{s}{2}} \partial_t H_{\frac{s}{2}}) t^{1-s} \partial_t V_{\frac{s}{2}}) \,\mathrm{d}t\,\mathrm{d}x \\
&\quad = \frac{1}{s}\int_{\mathbb{G}\times\mathbb{R}^+} -t^{2-s}(\Delta_b U_{\frac{s}{2}} H_{\frac{s}{2}} - U_{\frac{s}{2}} \Delta_b H_{\frac{s}{2}})\partial_t V_{\frac{s}{2}} \\
&\qquad\quad - t^{2-s}(\partial_t U_{\frac{s}{2}} H_{\frac{s}{2}} - U_{\frac{s}{2}} \partial_t H_{\frac{s}{2}}) \Delta_b V_{\frac{s}{2}} \,\mathrm{d}t\,\mathrm{d}x \\
&\quad = -\int_{\mathbb{G}\times\mathbb{R}^+} t^{2-s}(\Delta_b U_{\frac{s}{2}} H_{\frac{s}{2}} - U_{\frac{s}{2}} \Delta_b H_{\frac{s}{2}})\partial_t V_{\frac{s}{2}} \\
&\qquad\quad + t^{2-s}(\partial_t \nabla_{\mathbb{G}} U_{\frac{s}{2}} H_{\frac{s}{2}} + \partial_t U_{\frac{s}{2}} \nabla_{\mathbb{G}} H_{\frac{s}{2}}) \nabla_{\mathbb{G}} V_{\frac{s}{2}} \\
&\qquad\quad - t^{2-s}(\nabla_{\mathbb{G}} U_{\frac{s}{2}} \partial_t H_{\frac{s}{2}} - U_{\frac{s}{2}} \partial_t \nabla_{\mathbb{G}} H_{\frac{s}{2}}) \nabla_{\mathbb{G}} V_{\frac{s}{2}} \,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{6.2}$$

The first term can be bounded easily as in Theorem 6.6. For the last two terms, we also have the right bound since $s < 1 + 2\alpha = 1 + s$. □

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